REPORT RESUMES

ED 003 371 24

A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE. (THE MADISON PROJECT)

BY- DAVIS, ROBERT B.
SYRACUSE UNIV., N.Y.

REPORT NUMBER CRP-D-093 PUB DATE 65
WEBSTER COLL., WEBSTERS GROVE, MO.
EDRS PRICE MF-$0.27 HC-$6.08 152P.

DESCRIPTORS- *MATHEMATICS CURRICULUM, MATHEMATICS, *MODERN MATHEMATICS, TEACHING METHODS, CLASSROOM ENVIRONMENT, DEMONSTRATION PROJECTS, *ELEMENTARY EDUCATION, *CURRICULUM ENRICHMENT, MASS MEDIA, TAPE RECORDINGS, FILMS, TEACHER EDUCATION, *EDUCATIONAL THEORIES, SYRACUSE, NEW YORK, WEBSTER GROVES, MISSOURI, MADISON PROJECT

THIS REPORT COVERS THE THIRD AND FOURTH YEARS OF THE 5-YEAR "MADISON PROJECT" DESIGNED TO DEMONSTRATE AN UPDATING PROCESS FOR THE SCHOOL MATHEMATICS CURRICULUM IN ELEMENTARY EDUCATION. THE ATTEMPT WAS NOT ONLY TO BROADEN THE OVERALL CURRICULUM FROM GRADES 2 THROUGH 8 BY ADDING PORTIONS AND ASPECTS OF SUCH DISCIPLINES AS ALGEBRA AND COORDINATE GEOMETRY, BUT TO ACCOMPLISH (1) DIVERSITY IN CLASSROOM ACTIVITY, (2) ACTIVE STUDENT PARTICIPATION, AND (3) CREATIVITY IN CLASSROOM EXPERIENCES. IT WAS DECIDED TO DEVELOP SMALL INSTRUCTIONAL UNITS FOR INSERTION INTO A PREEXISTING CURRICULUM, RATHER THAN TO PREPARE AN ENTIRELY NEW CURRICULUM. THE "MADISON PROJECT" PROCEDURE WAS PREPARED FOR MATHEMATICIANS, OR OTHERS FAMILIAR WITH MATHEMATICAL IDEAS AND WITH WORKING WITH CHILDREN, WHO DEVELOPED FULLY-STRUCTURED (BUT FLEXIBLE) LESSONS IN ACTUAL CLASSROOM ENVIRONMENTS. THESE LESSONS, THEN, WERE EITHER TAPE-RECORDED OR FILMED AND USED AS MEANS OF (1) REPORTING TO THE TEACHING AND MATHEMATICS PROFESSIONS AND (2) TEACHER TRAINING. ANALYTICAL DESCRIPTIONS OF THE PROJECT METHODS AND MATERIALS WERE ACCUMULATED INTO A "THEORY OF INSTRUCTION" FORMAT. PARTICIPATING TEACHERS, GENERALLY, GAVE ENTHUSIASTIC RESPONSES TO THIS "THEORY OF INSTRUCTION," AS IT APPEARED TO BE IN A FORM THEY FOUND USEFUL AND RELEVANT TO THEIR PROBLEMS AND GOALS. IT WAS DETERMINED THAT ELEMENTARY STUDENTS (AT LEAST IN THE HIGHER ABILITY GROUPINGS) CAN LEARN FAR MORE MATHEMATICS THAN HAS USUALLY BEEN THOUGHT POSSIBLE. (JH)

U. S. DEPARTMENT OF HEALTH, EDUCATION AND WELFARE
Office of Education
This document has been reproduced exactly as received from the person or organization originating it. Points of view or opinions stated do not necessarily represent official Office of Education position or policy.

A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE. (THE MADISON PROJECT)

Cooperative Research Project No. D-093

Robert B. Davis

Syracuse University
Syracuse,
New York 13210

Webster College
St. Louis,
Missouri 63119

1965

The research reported herein was supported through the Cooperative Research Program of the Office of Education, U. S. Department of Health, Education and Welfare.

ED 003371

TABLE OF CONTENTS

Page No.

ABSTRACT iv

I. PROCEDURE, AND STATEMENT OF A PRIORI OBJECTIVES 1

Devising "Creative Learning Experiences." 3

Curriculum Planning. 3

Some Attributes of "Creative Learning Experiences." 4

Helping Teachers to Teach the "Creative Learning Experience" Lessons. 5

Provision for Continuing Curriculum Evolution -- The Decision to Develop a Supplementary Program. 8

II. MATHEMATICAL TOPICS 9

III. BRIEF HISTORY OF THE MADISON PROJECT 20

IV. OBJECTIVES 26

Objectives for Student Learning and Student Growth -- "Cognitive" or "Intellectual" Objectives. 26

Objectives for Student Learning and Student Growth -- "General" or "Attitude" Objectives. 27

Objectives for Teacher Learning and Teacher Growth. 29

Objectives for the Evolution of the School Curriculum. 29

Objectives for Curriculum Theory. 31

Objectives for the Operation of the Madison Project Itself. 31

Page No.

V. FURTHER DISCUSSION OF PROCEDURE 33

Provision for Teacher Education and Curriculum Revision Within the School Situation. 33

Curriculum Evolution in Urban School Systems. 34

VI. RELATED RESEARCH AND CURRICULUM EFFORTS 40

Mathematical Ideas. 40

Specific Pedagogical Ideas. 42

Rationale Concerning Schools and Learning. 42

Rationale Concerning Curriculum Evolution and Administrative Procedures. 43

VII. RELATED PROGRAMS 44

VIII. MATERIALS PRODUCED 50

IX. SOME FAILURES 85

X. COLLECTING AND INTERPRETING DATA; USE OF FEEDBACK; EVALUATION 87

XI. CONCLUSIONS AND IMPLICATIONS 106

Remarks relating to the General Idea of Progress. 106

Remarks Relating Primarily to Procedure. 107

Difficulties. 108

Remarks Concerning Unsettled Questions. 110

Some Conclusions. 110

Page No.

XII. BIBLIOGRAPHY 113

A. Publications by the Madison Project, or relevant publications by individuals associated with the Project. 113

B. Publications which refer to the Madison Project. 118

C. Publications which do not refer specifically to the Madison Project, but do present a rationale, or the results of studies, that are generally similar to or supportive of the Project's rationale and methods. 121

D. General publications which have a broad relevance to the Project's activities. 127

ABSTRACT

A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE.
(THE MADISON PROJECT)

Robert B. Davis

Syracuse University
Syracuse, New York 13210

Cooperative Research Project No. D-093

September 18, 1963 to September 17, 1965

Note

The present report covers the third and fourth years of a proposed 5 year "demonstration." This report bears the same title as two earlier reports; they can be distinguished not by title but by dates:

Report of the first year	dated 1962
Report of the second year	dated 1963
Report of the third and fourth years	dated 1965

Matters discussed extensively in these earlier reports are not duplicated in the present report, although in most cases they are still relevant. The reader is referred to the two previous reports.

BACKGROUND

The recent past has witnessed several significant efforts at modernizing the school curriculum, perhaps especially in mathematics and in science. Notwithstanding these efforts, the curriculum shows little tendency to change; in William Alexander's phrase, it has become "frozen," and it is not easily "unfrozen." The present project attempts to inject a systematic evolutionary pressure on the curriculum, in order to induce continuing growth of the curriculum.

OBJECTIVES

A. Objectives for the curriculum.

1. The demonstration has sought to broaden the curriculum by including important portions and aspects of mathematics ordinarily omitted, especially basic concepts of algebra and coordinate geometry in grades 2 through 8; mathematical logic, and matrix alge-

bra, in grades 5 through 9; combinations of science and mathematics in grades 5 through 9; and the theory of limits of infinite sequences, in grade 9.

2. The demonstration has sought to introduce a greater diversity of types of activities into the school mathematics program: in addition to traditional reading and writing of mathematics, the demonstration has added laboratory work with physical apparatus, seminar-type discussions, small-group "committee" work, and individualized instruction.

3. The demonstration has sought to give the student a more active role in mathematics experiences, to cast him less in the passive role of reader or listener, and more often in the role of experimenter or innovator or decision maker.

4. The demonstration has sought to identify various basic unifying concepts in mathematics (such as variable, function, Cartesian coordinates, matrices, isomorphism, etc.) that have been conspicuous by their absence, and to add them to the curriculum.

5. Considering that the traditional mathematics curriculum consistently omitted adequate "readiness building" prior to the introduction of new ideas, and was therefore bound to the level of rote instruction and imitation, the demonstration has given special importance to careful readiness building in advance of the appearance of new abstractions.

6. The demonstration has sought to develop a more creative flavor in the child's experiences with mathematics.

7. Considering that the traditional curriculum was regularly lacking in vitality and relevance, the demonstration has sought to increase the vitality and relevance of the student's experiences with mathematics, and thereby to increase the student's sense of commitment to school and to intellectual matters in general.

B. Objectives for the demonstration itself.

1. To work with selected schools and school systems in providing students with ex-

periences that accurately reflect a valid interpretation of "good mathematical thinking";

2. To make extensive collection of data that will serve to document, as specifically as possible, the nature of these experiences;

3. To achieve the most effective possible "quality control" in assuring the relevance and effectiveness of these experiences;

4. To integrate this "demonstration" as effectively as possible into the Madison Project's program of in-service teacher training, pre-service teacher education, and development of curriculum materials;

5. To report as effectively as possible to the profession and to the academic community: that is, to teachers, principals, supervisors, other educators, mathematicians, psychologists, scientists, board of education members, parents, etc. (It is assumed that this communication will be in the nature of dialogue, in which the Project will receive valuable counsel from the academic community, particularly concerning the effectiveness of Project materials and the general appropriateness of the learning experiences.)

PROCEDURE

1. Considering that curriculum evolution should be a continuous and on-going activity, the demonstration has chosen to develop small units for insertion into a pre-existing curriculum, rather than to prepare an entirely new curriculum.

Many schools introduce these units initially as a kind of supplementary or enrichment program, until, over a period of time, the units gradually become assimilated into the standard curriculum.

2. Tradirional teacher education has often consisted of introducing the teacher to

new mathematical ideas, thereafter leaving the teacher the task of devising appropriate ways for children to learn these ideas. Given that the teacher himself usually did not feel fully at home with the ideas, this was a procedure which did not often succeed.

The Madison Project procedure, by contrast, is for Ph.D. mathematicians, or others who feel entirely at home with the mathematical ideas and also with children, to work in classrooms and develop fully-structured (but flexible) actual classroom lessons. These lessons are then transmitted by observation of demonstration classes, or by viewing films which show actual classroom lessons. In some cases the films make it possible to observe every consecutive mathematics lesson for as long as a week; in other cases they show the progress of the same children at intervals over a span as long as five years. The films, of course, are intended to be viewed by teachers, not by children. (The development of these lessons, and the making of films, has been sponsored by a variety of different agencies, including the Marcel Holzer Foundation, the Alfred P. Sloan Foundation, the Ford Foundation, the National Science Foundation, Syracuse University, Webster College, Notre Dame University, and various participating school systems themselves.)

Operationally, the procedure has consisted of:

3. Developing an organizational structure between the Project and the participating school systems that provides for mathematically and pedagogically competent personnel to teach or to supervise work in each participating school;

4. Accumulating a library of tape-recordings of actual classroom lessons;

5. Accumulating a library of 16 mm. sound motion pictures recording actual classroom lessons, as a means of preserving the fullest possible record of the nature of the classroom experiences that are provided for the children (the cost of this has been met mainly

by a group of industries and trade unions in greater St. Louis, and by several grants from the National Science Foundation;

6. Using the films just mentioned as a means of reporting to the teaching and mathematics professions;

7. Using the films as a means of teacher training;

8. Extensive consulting with professional leaders as a guarantee of the relevance and appropriateness of the learning experiences provided by the Project.

9. Accumulating analytical descriptions of the Project's methods and materials in the form of a gradually growing "theory of instruction."

RESULTS

In this, and in the following section on Conclusions and Implications, it should be emphasized that the present demonstration is an effort at producing curriculum evolution in actual school situations; it is not a laboratory study, nor a "proof" of any particular thesis. Hence, its statements concerning "results" and "conclusions" are reports based on several years of experience in actual school operations. They do not "prove" anything and are not intended to do so.[1] The work reported here, however, should be of great value to anyone who is seeking to achieve objectives similar to those of this Project.

1. It has been possible to separate the lessons developed by the Project into two groups: those which are "stable," and can be taught by many teachers to many students with reasonable (though not perfect) reliability, and a second group of lessons that have not yet been cast in stable and reliable form, and may in fact never achieve stable and

---

[1] Michael Polanyi, Personal Knowledge, University of Chicago Press, 1958, and Carl R. Rogers, On Becoming a Person, Houghton Mifflin Co., Boston, 1961.

reliable form. A major question in the "stability" of a lesson is the extent to which appropriate teacher aids are available.

2. Although the Madison Project's methods and materials have been largely developed in culturally privileged suburban schools, it has turned out that the "stable" lessons, at least, can be used satisfactorily with culturally-deprived urban children, and can be taught successfully in typical urban settings.

3. The approach of creating a set of fully-developed classroom lessons, and passing these on to teachers via films showing actual classroom lessons, is a very powerful one.

4. Judging from the enthusiastic response of teachers and college faculty members, the accumulating "theory of instruction" appears to be cast in a form that teachers find useful and relevant to their problems and their goals.

5. Unfortunately, some aspects of mathematics and science in grades 3 - 8 tend to demand a great deal of teachers, especially in regard to the need for an open-ended, tentative approach (couched in what Schwab has called "a rhetoric of inquiry" rather than "a rhetoric of conclusions"), and in regard to the need for the teacher to be able to talk easily about mathematics without losing interaction contact with the child. These problems suggest clear needs in the future education of elementary school teachers.

CONCLUSIONS AND IMPLICATIONS

1. The highest 30%, or thereabout, of suburban children (in communities such as Weston, Connecticut) can learn far more mathematics in grades 2 through 9 than has usually been thought possible. The children appear to prefer such a highly enriched mathematics curriculum to a less enriched one. (To see what this means in detail, view the films entitled: A Lesson with Second Graders, Second Lesson, Graphing a Parabola, Graphing an Ellipse, Report to the Miami Meeting, Matrices, Solving Equations with Matrices, and

Creative Learning Experiences, among others.)

2. Mathematics can hold much more interest for many more students than has usually been realized, provided varied facets of mathematics are considered, and appropriately varied activities are included in the student's experiences.

3. Mathematics appears to have educative value in helping the student to learn about himself in relation to intellectual matters and to school in general, provided the mathematics has the appropriate relevance, vitality, and unequivocal honesty.

4. Curriculum revision efforts are possible even in cities the size of Chicago and St. Louis, provided suitable administrative arrangements are made.

5. Curriculum evolution can be influenced by the preparation of small units, for insertion into a pre-existing curriculum.

6. Curriculum evolution at the school level needs to become a significant part of the school's normal operating program. There is no satisfactory solution to on-going curriculum revision that is wholly based upon developing curricula outside the school and "importing" the curricula via new textbook adoptions.

7. Ordinary academic institutions (such as colleges and universities, state departments of education, textbook publishers, etc.) cannot provide for a vigorous high-quality program of curriculum evolution, at least not within their usual present modes of operation. Special action-oriented organizations are required; probably the best example of what is needed is Educational Services Incorporated, of Watertown, Massachusetts.

8. Even despite a large accumulation of studies of various sorts, the important matters of classroom social organization, "classroom ecology," and even more broadly, "school ecology" have not been described well enough to provide guidance to the curriculum innovator. As a result, curriculum innovation is guided by tradition more than is desirable.

9. The planning, as well as the actual operation, of possible school organizations, is restricted to an undesirably small range of possible organizational forms. There is in fact surprisingly little real variety, even though existing forms appear to embody severe limitations, and to function at a mediocre level, at best.

10. If we go somewhat beyond the Madison Project itself, and consider various participating programs (as we shall do in this item, and in items 11, 12, and 13 immediately below), we can cite the Webster College undergraduate teacher education program, which has been in operation for 4½ years, as evidence that a high-quality program to educate elementary school mathematics specialists along strictly modern lines (and including the study of 30 credits of college-level mathematics) is feasible and can even be conspicuously successful. This program shows clearly that the full range of possible teacher education programs has certainly not been explored; most present programs exhibit extremely little variation from one another, and leave important needs entirely unmet.

11. The example of the Webster College MAT program shows clearly that a program of graduate studies designed to enable experienced elementary school teachers to become mathematics specialists (by the study of 30 credits of classroom-related modern mathematics) is feasible and can operate with conspicuous success. Here, again, the Webster College program emphasizes the fact that most existing programs for graduate study by elementary school teachers exhibit too little variety, and leave important needs entirely unmet.

12. The high school mathematics program at Nova High School, in Fort Lauderdale, Florida, developed by Burt Kaufman, shows clearly that the more able mathematics students can learn nearly the entire traditional high school mathematics program on their own, outside of school, and can devote their school efforts to the study of what has previously

been thought to be college-level mathematics. Here, again, one sees that the diversity of school programs has not been very great, and has hardly explored the range of possible programs.

13. Somewhat more conservatively, but not less importantly, the student teaching program created by Adelyn Muller in suburban Kansas City shows that it is possible to combine self-contained classroom student teaching experiences with "visiting-specialist" student teaching experiences to develop a new type of program that meets some important modern needs.

14. Curriculum innovation efforts make it clear that we must rethink the relation of art and science in connection with curriculum and instruction. The science of physics could not progress far beyond "the state of the art" of equipment technology, computer technology, etc. The "science" of Marie Curie's chemistry could not have advanced far without her proficiency in the art of cleaning glassware, etc. To attempt "scientific" studies of education based upon a mediocre level of the art of teaching can indicate only what is possible within the context of a mediocre level of the art of teaching, and this can never be very much. Nor is it for any reason necessary to assume that reality constraints make a mediocre level of the art of teaching inevitable. Determined and inspired efforts can result in far higher levels of the art of teaching, and no significant advance in curriculum and instruction is possible without this prerequisite.

15. Films showing actual classroom lessons can become one of the most powerful of all the tools available for the improvement of education.

BIBLIOGRAPHY

There are 291 references listed in the final report.

PUBLICATIONS

1. J. Robert Cleary, "A Study of Test Performance in Two Madison Project Schools and One Central School," 1965, available from the Madison Project.
2. Donald Cohen, "A Lesson on Absolute Value," The Arithmetic Teacher, Vol. 11, No. 8 (December, 1964), pp. 561 - 562.
3. Robert B. Davis, A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project, 1962, report submitted to the Commissioner of Education, U. S. Department of Health, Education and Welfare.
4. Robert B. Davis, "Report on the Madison Project," Science Education News (December, 1962), pp. 15 - 16 (a publication of the American Association for the Advancement of Science). (Sponsored jointly with NSF)
5. Robert B. Davis, A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project, 1963, report submitted to the Commissioner of Education, U. S. Department of Health, Education and Welfare.
6. Robert B. Davis, "The Evolution of School Mathematics," Journal of Research in Science Teaching, Vol. 1 (1963), pp. 260 - 264. (Sponsored jointly with NSF)
7. Robert B. Davis, "The Syracuse University - Webster College Madison Project," in An Analysis of New Mathematics Programs, edited by Philip Peak, et al. NCTM (1963), pp 16 - 19; 20 - 21. (Sponsored jointly with NSF)
8. Robert B. Davis, "Mathematical Thought and The Nature of Learning: The Madison Project View," Frontiers of Education. Report of the Twenty-Seventh Educational Conference sponsored by the Educational Records Bureau. American Council on Education, Educational Records Bureau, 21 Audubon Ave., New York 32, New York (1963), pp. 79 - 83.

9. Robert B. Davis, "How Is The Madison Project Different?" Illinois Council of Teachers of Mathematics News Letter, Vol. XIV, No. 2 (May, 1963), pp. 7 - 8.

10. Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction, Journal of Research in Science Teaching, Vol. 2 (1964), pp. 146 - 162.

11. Robert B. Davis, "Undergraduate College Education in Relation to the 'New Curriculum' Projects," Current Issues in Higher Education (1964), pp. 153 - 155.

12. Robert B. Davis, "Report on the Syracuse University - Webster College Madison Project," American Mathematical Monthly, Vol. 71, No. 3 (March, 1964) pp. 306 - 308. (Sponsored jointly with NSF)

13. Robert B. Davis, "Some Remarks on a Theory of Instruction," Piaget Re-discovered, Edited by Verne N. Rockcastle and Richard E. Ripple. A report of the Conference on Cognitive Studies and Curriculum Development (March, 1964). School of Education, Cornell University, pp. 134 - 138.

14. Robert B. Davis, Experimental Course Report/Grade Nine (Report No. 1, June, 1964), available from the Madison Project.

15. Robert B. Davis, "Guidelines for the Assessment, Initiation, and Reassessment of Curriculum Programs" (paper presented at the Chicago ASCD meeting, Fall, 1964).

16. Robert B. Davis, "Goals for School Mathematics: The Madison Project View, Journal of Research in Science Teaching, Vol. 2, Issue 4 (December, 1964), pp. 309 - 315. (Sponsored jointly with NSF)

17. Robert B. Davis, "Recent Activities of the Madison Project," American Mathematical Monthly (January, 1965). (Sponsored jointly with NSF)

18. Doris Diamont Machtinger, Experimental Course Report/Kindergarten. (Report No. 2, July, 1965). The Madison Project (in preparation).

19. Adelyn Muller, "A Report on Student Teaching in the District 49 Public Schools (Overland Park, Kansas)." March 16, 1965. (Available from the Madison Project.)

A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE.
(THE MADISON PROJECT)

I. PROCEDURE, AND STATEMENT OF A PRIORI OBJECTIVES

The Madison Project of Syracuse University and Webster College is concerned with the evolution of the school mathematics curriculum. The evidence of the first half of the twentieth century shows that the curriculum easily becomes "frozen" in a rigid pattern that fails to match the progressive changes in our society. Change in the curriculum can be produced only by the deliberate investment of time, money, and effort, in a coordinated way and on a suitably large scale. Appropriate change requires also careful and intelligent planning, and the rapid utilization of feedback.[1]

This "rapid use of feedback" is, in fact, extremely important, since the Project operates on a rationale that might be described as: "do something," then "discuss, analyze, and criticize what was done," then, "using the fruits of this discussion and analysis, do something again, in a modified fashion," and so on.

This rationale is different from that whereby one plans first, and then executes according to that plan. While the "plan first" approach has a certain attractive simplicity, it is not, in fact, a powerful approach in curriculum work, since it deprives the teachers and the planners of any opportunities to learn as the work progresses. The point is of such basic importance, and the "plan first, then execute according to plan" rationale has become so firmly established in educational work, that further discussion is necessary, perhaps by way of an analogy. In the design and use of space rockets, two kinds of rockets are employed: those which are "purely ballistic," in the sense that they are aimed and fired, as one fires a bullet from a gun, and those which can be "flown," in the sense that they can be controlled after launching. In the "purely ballistic" type, the smallest of initial errors made at launching becomes tremendously magnified in (perhaps) several million

---

(The bibliography at the end of this report is divided into four sections, labeled A,B,C, and D. In reference to the bibliography, the capital letter preceding the date indicates the section of the bibliography in which the reference appears; the rest of the reference follows a form in common use at the present time.)

[1] Cf. for example: Robert S. Gilchrist, et al., Using Current Curriculum Developments, Association for Supervision and Curriculum Development (1963); Frazier (D1964); Alexander (C1964); Glenys G. Unruh, editor, New Curriculum Developments, ASCD, 1965; Goodlad (B1964); Fletcher (C1964); Bruner (C1963b).

miles of subsequent flight, but once the missile is launched, no further control is possible. The same thing occurs whenever a bullet misses its target. By contrast, if the rocket is capable of later adjustments, initial errors can be corrected at later points in the flight, or the missile can "home in" on its target. Driving an automobile is a similar case in point: to "fire" an automobile like a bullet, with its entire course and subsequent movements fully determined at the start of a journey, would be unthinkable. The only possible approach is to utilize feedback as one drives along, adjusting the motion of the car in whatever ways may be necessary.

The Project's basic rationale depends upon this same sort of continual utilization of feedback: evidence is continually sought as to the course of the effort thus far, and modifications are made as quickly as the need for them can be identified. The specific applications of this principle will become clear in the following pages.

At the outset of a curriculum evolution effort, one cannot fully specify what needs to be changed, nor in what direction, nor to what extent, nor by what means. The "exploration" procedure used must resemble the physicists' "method of perturbations" or "successive approximations."

As a "zero-th approximation," the Project's initial a priori objective was to try to discover the best possible experiences with mathematics which can be provided for children, and to devise methods to enable as many children as possible to have these experiences. Obviously, with further explorations, these objectives can be structured more fully, and on a more detailed tactical level.

One might say that the basic assumptions underlying the Project's curriculum effort are these:

i) a group of serious mathematicians, psychologists, psychiatrists, physicists, school personnel, and others, working together, can devise experiences in mathematics that will be more appropriate for most children than those traditionally encountered, and can judge this appropriateness with reasonable agreement;

ii) that these same types of professional people can study and discuss the progress made, and guide future efforts toward yet more satisfactory experiences;

iii) that, by articulating these discussions, these same types of professional people can make some sort of contribution toward a "theory of instruction."

With the passage of time, these broad general statements have been replaced with increasingly detailed statements, as will be seen below.

The basic procedures of Project activities might be described as follows:

1. Devising "Creative Learning Experiences." After a list of basic mathematical concepts has been compiled (a process to be described presently), then actual classroom experiences are worked out by repeated classroom trials. In the early stage of developing new materials, the classroom lessons are taught by Ph. D. mathematicians, or by other quite sophisticated professionals. The lessons gradually assume more definite shape as a result of repeated trials with different classes. When a lesson is in moderately definite form, it is added to the list of the Project's "tentative" materials, and usually receives wider trials. If the lesson reaches the stage of reasonably definitive form, backed up by a consensus as to its desirability, it is added to the list of the Project's "stable" materials, and is made generally available to anyone who is interested.

Because these "lessons" often take novel forms, frequently somewhat at variance with many of the traditional forms of school "lessons," it has seemed desirable to give them a distinctive name, and so they have been labeled "creative learning experiences."

2. Curriculum Planning. One must evidently begin with a list of fundamental mathematical concepts. For the Madison Project, at present, this list includes:

the concept of number
the process of counting
the concept of addition
the concept of multiplication
the concept of division
the concept of subtraction
place value numerals
the number line
the arithmetic of signed numbers
the process of measurement
measurement error
Cartesian coordinates
the arithmetic of rational numbers
variables and open sentences
functions
identities
implication
contradiction
uniqueness
axioms and theorems
"derivations" (i.e., proofs of algebraic theorems)
simultaneous equations
the concept of angle
the concept of area and volume
linear graphs (including slope and intercept)
the logic of generalizing from instances vs. the logic of deductive proof
graphs of conic sections
linearity vs. non-linearity; ratio and proportion

trigonometric functions
exponential functions
linear, quadratic, and other polynomial functions
finite difference methods for identifying polynomial or exponential functions
empirically-determined functions
average, variance, and similar statistical notions
the basic algebra of matrices
truth tables, inference schemes, and similar notions from mathematical logic
rate of change
graphical integration and differentiation
displacement, velocity, and acceleration
limit of monotonic sequence
limit of non-monotonic sequence
general form of an equation, and its general solution

among others.

As such a list is gradually assembled, it is also necessary to begin allocating topics to approximate grade levels. To some extent there are natural constraints within the material itself -- ratio and proportion should probably be familiar before the student begins the study of trigonometric functions; some arithmetic of signed numbers is necessary before the student attempts to graph a circle. Nonetheless, the constraints imposed by this sort of "necessary prerequisite" are far less severe than is usually imagined. Indeed, David Page has pointed out that for every reason why topic A should precede topic B in the child's course of study, there is usually an equally urgent reason why topic B should precede topic A. This appears to be surprisingly true, particularly when one examines the curriculum with imagination.

The assignment of topics to approximate grade levels depends not so much upon the constraints imposed by necessary prerequisites, but rather primarily upon the question of whether suitable lessons can be devised to allow the child to learn these concepts. For example, despite several attempts, the Project has not been successful in devising lessons on the concept of limit of a sequence that are appropriate for sixth graders. This brings us back, again, to the nature of the "creative learning experiences" -- that is to say, the actual classroom lessons themselves.[1]

3. Some Attributes of "Creative Learning Experiences." The "Ph. D. mathematicians or other sophisticated professionals" who go into the classroom to work out suitable lessons are guided by a set of criteria, approximately as follows:

i) The lessons should be directly related to one or more of the "fundamental mathematical concepts" on the list above, with a minimum of extraneous complication. Lessons are never considered merely because they involve an "interesting puzzle" or a "nice thing to do with children." They must relate directly to items on the list of fundamental mathematical concepts.

---

[1] The freedom to permute the order of occurrence of different topics in the curriculum must not be exaggerated. While appropriate order is not so originally determined as was once assumed, nonetheless some order-sequences are far more "natural" and satisfactory than others.

ii) The lessons must provide an active role for the children, and, as much as possible, an autonomous decision-making role for each child. "Passive" or "captive" experiences such as reading, listening to a lecture, or "reciting" in a rote fashion are avoided. (Obviously, this is a hard requirement to state in words, for what is a passive experience for one child may be an active experience for another. Certainly, for the "good math student," reading and listening to lectures are active experiences, for he is ever inquiring into "what next?" and "why?", etc.)

iii) The lessons should often (though not necessarily always) provide opportunities for the children to make interesting original discoveries.

iv) The lessons should usually present the mathematical ideas within a natural framework, so that each mathematical concept or technique appears as a reasonable answer to a reasonable question.

v) The lessons should be appropriate for these particular children, taking account of their age, their general nature and apparent needs, their background, their previous experiences with mathematics, and so on.

These criteria are discussed at greater length elsewhere.[1] What is important is that a rationale for decision-making does exist, is being elaborated and continually refined, and is being communicated to the academic community as effectively as possible.

4. Helping Teachers to Teach the "Creative Learning Experience" Lessons. At this point, then, the Project has run informal classroom trials until it is in possession of actual classroom lessons that enable children to learn various appropriate concepts and techniques in mathematics.

What comes next?

Evidently, as more widespread trials indicate that the lessons in question are suitable for many children and can be taught by many teachers, it becomes desirable to shift the emphasis to the question: how can we help teachers to teach these lessons effectively?

---

[1] Cf. Davis (A1962c) and Davis (A1964c).

Two remarks need to be made: first, notice that the teacher-education task confronting the Project is "inside out," or the reverse of the usual problem. Most other attempts depend upon teaching some new mathematical ideas to teachers; then, while the idea is still unfamiliar and uncomfortable to the teacher, the teacher has the task of devising classroom lessons that will enable children to learn these ideas clearly and easily. This is a nearly impossible task for most teachers.

The Madison Project's approach is the reverse of this. Actual classroom lessons for use with children have been devised and carefully polished. They work. These lessons then form the starting point for work with teachers. A deeper probing into mathematical ideas comes out as the teachers discuss these actual lessons. (These "actual classroom lessons" are ordinarily presented to the teachers either by using a demonstration class of children, or else via films showing actual classroom lessons.)

We shall refer to this in the future as the Project's "inside-out" approach to teacher education.

Second, it should be noticed that there are no sharp dividing lines between various stages. There should be none, since each stage is dependent upon the others, and they are dependent upon it.

For example: at what stage shall the Project make a film showing a particular classroom lesson? Evidently, not while the lesson is still highly tentative, for making films is expensive. Yet, once the film is made, it will be shown to a large number of mathematicians, psychologists, teachers, and others, and these professional viewers will make suggestions and express judgements. Thus, the film cannot be made "after the lesson is in final form," for the very act of making the film, and allowing people to view it, will tend to modify the lesson itself.

Similarly, at what stage will the lesson be communicated to a wider audience of teachers? Evidently, not while the lesson is highly tentative, for such a procedure would be inefficient and confusing. Yet, wider trials of the lesson cannot be made until a larger number of teachers understand the lesson. (Furthermore, it is desirable, at least in the case of more sophisticated teachers, to allow them to participate in the task of refining the lesson, and putting it in "polished" form. This process is highly instructive.)

One consequence of this interdependence of various stages and various tasks is a corresponding interdependence of work supported by different sponsors. In general, the financial support of Madison Project work has been as follows:

i) Early development of "creative learning experiences" (1957) -- not sponsored.

ii) Further development of "creative learning experiences" (1958 - 1960) -- financial support from Marcel Holzer Foundation and Alfred P. Sloan Foundation.

iii) Early collaboration with schools, in order to help modernize the school curriculum (1957 - 1960) -- financial support from Syracuse University, Marcel Holzer Foundation, Alfred P. Sloan Foundation, and various participating school systems.

iv) Recording of actual classroom lessons on audio tape (1959 - 1960) -- financial support from Alfred P. Sloan Foundation, and various participating school systems.

v) Teacher education via summer institutes and in-service institutes (1958 - present) -- National Science Foundation, Institutes Section.

vi) Pre-service (college undergraduate) teacher education at Webster College (1961 - present) -- financial support from Webster College and the Ford Foundation.

vii) Later collaboration with schools, in order to modernize school curricula (1960-present) -- financial support from Division of Cooperative Research, United States Office of Education.

viii) Early recording of actual classroom lessons via video-tape and 16 mm. motion picture film (1960 - 1961) -- financial support from Webster College, from Notre Dame University, and from a group of industrial sponsors in the greater St. Louis area (Monsanto Chemical, etc.)

ix) Later recording of classroom lessons on film (1961 - present) -- financial support mainly from National Science Foundation, Course Content Improvement Section, with some additional funds from the U.S. Office of Education.

x) Production of lessons combining mathematics and science (1963 - present) -- National Science Foundation.

xi) Exploratory Ninth-Grade Course (1963 - 1965) -- financial support from the National Science Foundation.

xii) Production of "packaged" teacher education materials (1963 - present) -- financial support mainly from National Science Foundation.

xiii) Teacher education via Webster College MAT program (1963 - present) -- Ford Foundation and Webster College.

xiv) "Big city" program for large scale teacher education (1963 - present) -- U.S. Office of Education, Chicago Board of Education, St. Louis Public Schools, San Diego County Board of Education.

xv) Study of tape-recordings of actual classroom lessons, and accumulation of a growing "theory of instruction" (1961 - present) -- U. S. Office of Education.

In referring to some of these activities we have gotten somewhat ahead of our story, but the pattern of interdependence of activities (and, consequently, of sponsored programs) should be clear.

The use of the word "lessons" in the preceding pages may give the wrong impression. The Madison Project is not concerned primarily with a single 30 minute (or one hour) experience. In most cases, a mathematical topic is developed carefully over a sequence of lessons. The words "lessons" or "creative learning experiences" or "units" are used somewhat interchangeably in this report. Distinctions are at most a matter of shifting emphasis -- but in every case we are talking about the actual experiences that real children have in real schools, usually covering an interval of days or weeks, and integrated into an over-all program that progresses from year to year.

5. Provision for Continuing Curriculum Evolution--The Decision to Develop a Supplementary Program. Since the Project is concerned with the on-going task of continually revising the school curriculum, it employs an approach that injects new material, or modified "old" material, into small localized "pieces" of the school curriculum. Hopefully, this is gradually assimilated into the over-all curriculum, and, on present evidence, this approach works quite satisfactorily.

It is important to remember, then, that at any given time the Project focuses its efforts on a few small pieces of the over-all curriculum. In this sense, the Project is concerned with a "supplementary" program. At no single time does it attempt to prepare all necessary materials for a complete K - 12 program. Nonetheless, over a period of years broad portions of the school curriculum should evidence the impact of these repeated localized injections of new ideas and new teaching procedures.

II. MATHEMATICAL TOPICS

Although it is again getting ahead of our story to mention the materials by means of which the Project attempts to communicate its "creative learning experiences" to larger numbers of teachers, let us note that these materials involve both printed matter and also films showing actual classroom lessons. Where these materials are not yet available, communication is "face to face" by word of mouth, through workshops, in-service courses, summer institutes, or the programs at Webster College.

Before proceeding further, let us consider the list of mathematical topics with which the Project is primarily concerned at the present time.

As discussed earlier, these topics are divided into "exploratory" topics--i.e, topics where there is not at present a firm consensus on the appropriateness of this topic at this grade level--and "stable" topics--i.e., topics well-accepted, well-tested, and known to be appropriate for many children and many teachers at the grade level indicated.

As far as grade levels are concerned, the following listing distinguishes 4 levels:

primary grade level--i.e., nursery school, kindergarten, grades one and two, and possibly grade 3.

intermediate grade level--grades 4 through 6, inclusive, and possibly part of grade 3.

junior high school level--grades 7 and 8.

grade nine--obviously, grade 9 (except that the Madison Project program for grade 9 is more ambitious than is usual at this level--for example, it includes consideration of limits of infinite sequences--and hence in some schools part of this program might occur in grades 10 through 12).

The topics are as follows:

A. Primary Grades, "Stable" Topics:

1. Concepts of Numbers (for Small Positive Integers). Each child is provided with a pile of colored plastic washers. The teacher asks: "Show me what we mean by 'four'." The child responds by exhibiting four washers. Obvious variations and extensions are used. (WOM)[1]

---

[1] Designations of sources of these topics within Madison Project materials are as follows: WOM ("word of mouth") indicates a topic passed on by direct face-to-face contact with teachers, and not included in any present printed materials or films; "MLoT" refers to the book Matrices, Logic and other Topics: "Discovery" refers to the book Discovery in Mathematics.

2. Counting. Practice in counting out, say, 15 pebbles accurately (extended later to counting reliably as many as 200 or more pebbles, or dots on paper, etc.). Cf. the film entitled Subtraction and Division Using Beans (video-tape no. VT-38).

3. The Number Line. Assigning number names to points on the line, in the usual fashion at least for small positive integers, and perhaps also for zero, negative integers, and fractions. (Films: A Lesson with Second Graders; Graphs and Truth Sets; also, In-Service Course I.)

4. Measurement. Measuring length of table, etc., by using a tongue-depressor (etc.) as a unit. Consideration of cases where result "does not come out even." (Primarily WOM)

5. Introduction of Signed Numbers (Integers) by Means of the "Pebbles-in-the-Bag" Model. Cf. the film: A Lesson with Second Graders; also MLoT, Chapter 4; and In-Service Course I.

6. Variables and Open Sentences. Cf. the films: Graphs and Truth Sets, Open Sentences and the Number Line, and First Lesson; also, Discovery, Chapter 1; MLoT, Chapters 1, 2, and 3; and In-Service Course I.

7. Plotting Points in Cartesian Coordinates, Via a Modified Version of the Game Tic Tac Toe. Films: A Lesson with Second Graders, Graphs and Truth Sets; also MLoT, Chapter 2; In-Service Course I.

8. Graphs for Addition Facts. Cf. the film Graphs and Truth Sets.

9. Experience with Fractions, Using Kindergarten Blocks. As one example, 15 blocks are stacked up, and three children are each asked to "take one-third of them." Films: Experience with Fractions (video-tape no. VT-14), and also Experience with Fractions (video-tape no. VT-21).

10. Place-value Numerals, Using Beans and Tongue Depressors. Cf. the film: Operations of Arithmetic (video-tape no. VT-58).

11. What Do Mean By Addition? Each child has a stack of colored plastic washers. The teacher asks, "What do we mean by 'three plus four'?" Children give examples, using the washers. (WOM)

12. What Do We Mean By Multiplication? (With Plastic Washers). (Similar to No. 11 above; WOM)

13. What Do We Mean By Multiplication? (Dots on Paper). Many children use rectangular arrays of dots drawn on paper to answer the question. Second graders can often cope with problems such as 23 x 4. (WOM)

14. What's My Rule? This is a simplified version of "Guessing Functions." Cf. MLoT, Chapter 25; also the film: Guessing Functions. (Of course, for primary grades one uses only

"rules" of the form "whatever number you tell me, I add one and tell you the answer.")

15. Uses of Monotonicity. (E.g., for the equation $(3 \times \square) + 5 = 9$, 1 is too small, 2 is too large, etc.) Cf. Discovery, Chapter 22, and the film Solving Equations with the Number Line (video tape number VT-22).

16. Inequalities. (Cf. Discovery, Chapter 1.)

B. Primary Grades, Exploratory Topics:

(These topics have been tried by various Project teachers, but their suitability for primary grades is still in doubt, for various reasons.)

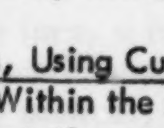
17. The "What Did I Change?" Game. Used particularly with culturally-deprived kindergarten children, the game employs wooden shapes (sphere, ellipsoid, cone, truncated cone, pyramid, cone with parabolic cut, cone with hyperbolic cut, cube, triangular prism, etc.) available from school supply companies. Four shapes are placed in view, and the children look at them carefully. Then an opaque cloth is used to screen them from view, and one form is removed and replaced by a new form. When the cloth screen is whisked away, the children are asked to tell which shape has been removed, and what new one added.

18. Various other games, somewhat similar to No. 17 above, can be played, using these same wooden shapes. One game is: "How Would I Make...?" For example, "How would I make a space rocket?" (Answer: "Put the cone on the end of the cylinder," or something similar.) For culturally-deprived kindergarten children, these games may help improve language usage, and give experience in careful observation, imaginative thinking, and paying attention, as well as building familiarity with geometric shapes.

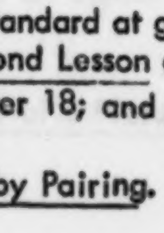
19. Experience with Fractions, Using Cuisenaire Rods. This topic follows the standard Cuisenaire-Gattegno presentation. Within the Madison Project program, it is standard at grade 4 or 5, but is still somewhat tentative at the primary grade level.

20. Identities. This topic is standard at grades 4 and 5, but is tentative at the primary grade level. Cf. the films: Second Lesson and Experiences with Identities (VT-22); also, Discovery, Chapter 5; MLoT, Chapter 18; and In-Service Course II.

21. Odd and Even Numbers, by Pairing. Pictures such as



and



are used to suggest the distinction between odd and even numbers (in the case above, 8 is even, and 7 is odd).

22. Following No. 21, Describe the Arithmetic of "Odds" and "Evens"

e.g., Even + Even = Even

Odd + Odd = Even

Odd + Even = Odd

etc.

23. Following No. 22, Give an Axiomatic Description of the Arithmetic of "Odds" and "Evens."

The axioms are:

i. 1 + 1 = 2

ii. 2 is even

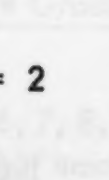
iii. even + 1 = odd

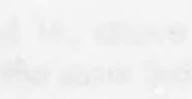
iv. every odd = even + 1

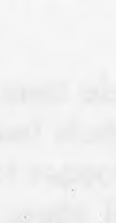
v. even + even = even ,

plus the commutative and associative laws, a law of "excluded middle," etc., which (surprisingly?) children use intuitively. Once the axioms are given, use them to prove various theorems. This topic looks dreadfully abstract, and the Project is presently unsure of its appropriateness at this grade level, but in trials thus far the response of first-grade children has been creative and enthusiastic to an almost unbelievable degree.

24. Games involving Rotations of a Picture, such as







25. Experience with Angles: Measuring with Pre-shaped Units.

26. What Do We Mean By Division?-- Using Beans or Pebbles. Cf. the film Subtraction and Division Using Beans (video-tape no. VT-38).

In addition, the Madison Project maintains close contact with similar exploratory efforts conducted by the Cambridge Conference (Educational Services Incorporated, or "ESI"), the Elementary Science Study (the "ESS" division of ESI), the Webster College Institute of Mathematics, Science and Arts ("WIMSA"), and other similar groups. The assessment of the long-range curricular value of some topics--the popular "attribute blocks" of Vygotsky, Dienes, and ESS might be a case in point--is extremely difficult. In general, the Madison Project is somewhat conservative about using materials or experiences whose real value is hard to ascertain. This problem appears to be especially acute at the level of nursery school, kindergarten and grades one and two. Indeed, systematic study of the value of pre-mathematical "readiness-building" experiences may be urgently needed (for example, in the case of the use of the Vygotsky "attribute blocks").

C. Intermediate Grades (roughly, grades 4, 5, and 6; some inclusion of third graders), Stable Topics:

Topics 3, 4, 5, 6, 7, 8, 15, and 16, above are used also in the intermediate grades. Obviously, this does not mean that the same individual student meets these topics again and again; these topics are used as necessary, and are not repeated when they are not needed. Many students enter the program at the intermediate grade level. For these "new" students, most of the topics listed will be new, and hence will be necessary at the intermediate grade level.

27. Guessing Functions. This topic is similar to No. 14 above, but is used in much more sophisticated form at the intermediate grade level. Cf. the films Guessing Functions, and A Week of Mathematical Exploration, Part II, Tuesday, Part III, Wednesday, and Part V, Friday; also MLoT, Chapters 25 and 26, and In-Service Course I.

28. Exponential Functions. Experience with functions of the form

$$2^{\square} = \triangle$$

$$3^{\square} = \triangle$$

$$3^{\square} + 1 = \triangle$$

etc. (WOM, plus audio tapes).

*29.[1] Classification of Functions as Linear, Quadratic, or Exponential, by Using Finite Difference Methods. (WOM, plus audio tapes)

*30. Empirically-determined Functions: The Tower Puzzle. Cf. the films The Tower Puzzle, and A Week of Mathematical Exploration, V, Friday; also MLoT, Chapter 27; and an audio tape from the Chicago workshop of 1965.

*31. Empirically-determined Functions: The Shuttle Puzzle. Generally similar to #30 above, but leads to a quadratic function. (WOM, plus audio tapes from the Chicago workshop of 1965.)

*32. Empirically-determined Functions Involving Measurements: Elasticity. This topic follows an approach developed by Professor Robert Karplus of the Berkeley physics department. Cf. the film Weights and Springs; also Discovery, Chapter 49; MLoT, Chapter 27; and In-Service Course I.

*33. Experience with Angles. Cf. the films Experience with Estimating and Measuring Angles (VT-16) and Experience with Angles (VT-29, Part II).

34. Experience with Fractions, Using Cuisenaire Rods. Follows the Cuisenaire-Gattegno presentation. This topic is, for the Madison Project, exploratory in the primary grades (that is, the Project is not yet convinced that the primary grades are, in 1965, the optimal placement of this topic), or else standard for grades 4 or 5. (WOM, or else refer to Cuisenaire materials.[2])

35. Experience with Surface Area and Volume, Using Cuisenaire Rods. Some of this work was developed by Professor David Page, of the University of Illinois (presently on leave and at ESI). Cf. Discovery: Text for Teachers, pp. 271 ff; also MLoT, Chapter 27. (Some of this work is presently unwritten, and transmitted WOM.)

36. Complete Arithmetic of Signed Numbers, Using "Postman" Model. Cf. the films: Postman Stories (VT-45); A Week of Mathematical Exploration, Part II, Tuesday; and Postman Stories and Graphing With Signed Numbers; also, Discovery, Chapters 12 and 14; MLoT, Chapters 5 and 6; and In-Service Course I.

---

[1] Topics marked with an asterisk are particularly suitable for individualized or small-group instruction. Cf. page 25.

[2] Available from the Cuisenaire Company of America, Inc., 9 Elm Avenue, Mt. Vernon, New York 10550.

37. Practice on Variables, etc., Using Quadratic Equations. Cf. the films First Lesson and Second Lesson; also, Discovery, Chapters 3, 5, 8, 13 and elsewhere; MLoT, Chapter 10; and In-Service Course I.

38 38. Linear Graphs. Cf. the films; Second Lesson, Experience with Linear Graphs (VT-23), Truth Tables and Graphs (VT-42); also, Discovery, Chapters 11, 15, 18, and elsewhere; MLoT, Chapter 3; and In-Service Course I.

39. Graphs of Conic Sections. Cf. the films Postman Stories, Circles and Parabolas, Graphing a Parabola, Graphing an Ellipse; also Discovery, Chapter 35, p. 73 (Student Book); and MLoT, Chapter 8.

40. Identities. Cf. the films Second Lesson and Accumulating a List of Identities (VT-18); also, Discovery, Chapter 5; MLoT, Chapter 18; and In-Service Course II.

41. Derivations. The Project has explored many alternative methods of writing out proofs of algebraic theorems, from an informal beginning up to a relatively formal method using "U.V." and "P.N." Cf. the films 7th Grade Derivations (VT-1,2) Super Identities (VT-59); Ninth Grade Algebra (VT 35 and 36); and Axioms and Theorems (VT-10); also Discovery, Chapters 28, 29, 30, 31, 32, and 33; MLoT, Chapters 19, 20, 21, 22, and 23; and Ninth Grade Report.

42. Equivalent Equations and Transform Operations on Equations. Cf. Discovery, Chapters 23, 24, and 25; and MLoT, Chapter 29.

43. Transform Operations on Inequalities. Cf. MLoT, Chapter 30.

44. General Form of Equations, and General Solution. Cf. the films 5th Grade Machines, Quadratic Formula -- First Beginning's (VT-3); Quadratic Formula -- Final Summary (VT-6); and Ninth Grade Algebra (VT-35 and 36); also, Discovery, Chapter 37, 38, and 41; and MLoT, Chapter 31, 32, 33, and 34.

45. Derivation of Geometric Mensuration Formulae. Cf. the film An Area of the Parallelogram Part 1 (VT-9); also, Discovery, Chapters 43 and 50; and "Geo-boards," item 55 below.

46. Simultaneous Equations. Cf. Discovery, Chapter 45.

47. Word Problems. Cf. the film Word Problems and the $\sqrt{2}$ (VT-9), Part 1, and Discovery, Chapters 44 and 46.

48. Average, Variance, Measurement Errors. Cf. the film Average and Variance and Units of Measurement, Distance and Angles (VT-49), Part II; also, MLoT, Chapter 17.

49. Experience with Implication, Contradiction, and Uniqueness. (The Game of Clues). This involves modification of David Page's "Hidden Numbers." Cf. the film Clues (VT-13) Part I and MLoT, Chapter 16.

50. Non-Standard Numerals and Non-Standard Algorithms. Cf. the films Kye's Arithmetic and Education Report: the New Math; also, MLoT, Chapter 7; and In-Service Course I.

51. Adding and Multiplying Matrices. Cf. the film Solving Equations With Matrices VT-13) Part II, MLoT, Chapter 37.

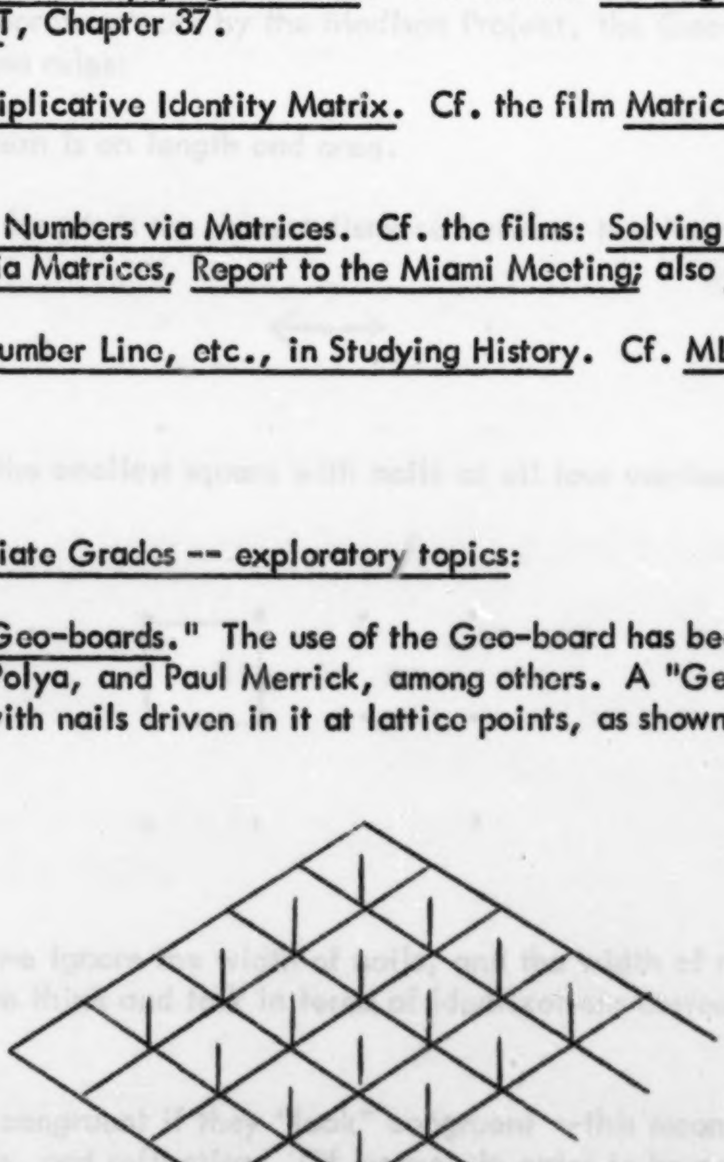
52. The Multiplicative Identity Matrix. Cf. the film Matrices; also MLoT, Chapter 38.

53. Complex Numbers via Matrices. Cf. the films: Solving Equations with Matrices, Complex Numbers via Matrices, Report to the Miami Meeting; also MLoT, Chapter 45.

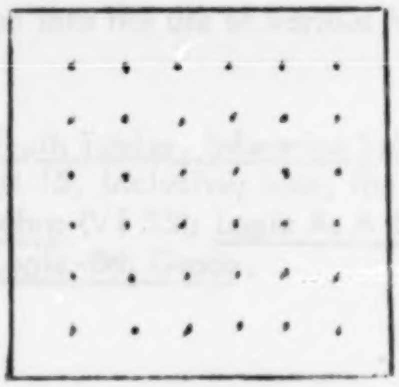
54. Use of Number Line, etc., in Studying History. Cf. MLoT, various chapters.

D. Intermediate Grades -- exploratory topics:

*55. Use of "Geo-boards." The use of the Geo-board has been pioneered by Caleb Gattegno, George Polya, and Paul Merrick, among others. A "Geo-board" is usually a board of plywood, with nails driven in it at lattice points, as shown:



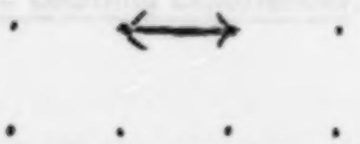
or diagrammatically,



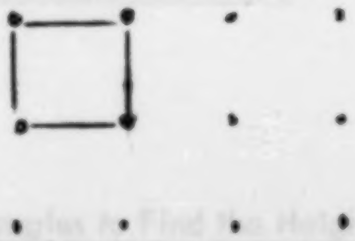
As used in exploratory work by the Madison Project, the Geo-board is a geometric device following these rules:

i) Major concern is on length and area.

ii) The unit of length is the shortest distance between two "consecutive" units:



and the unit area is the smallest square with nails at all four vertices:



iii) Obviously, we ignore the width of nails, and the width of rubber bands that we put over nails; that is, we think and talk in terms of idealizations corresponding to points, line segments, etc.

iv) Figures are congruent if they "look" congruent --this means, really, that we allow translations, rotations, and reflections. Of course, in order to be considered congruent, figures must have corresponding nails in precisely corresponding positions.

v) Arguments must be based on the above notion of congruence, and on the additive property of area. Formulas (such as $\frac{1}{2}$ bh) are not used.

vi) Gradual extension into the use of various formulas is made later on.

56. Logic, Including Truth Tables, Inference Schemes and Mappings of $\mathcal{U} \times \mathcal{U} \to \mathcal{U}$. Cf. MLoT, Chapter 11 through 15, inclusive; also, the films: Truth Tables and Inference Schemes; A Language for Algebra (VT 53); Logic As A Language For Algebra (VT 54); A Seminar (VT 56, 57); and Logic-8th Grade.

E. Junior High Grades (primarily grades 7 and 8); Stable Material.

All material above is used where necessary and appropriate in grades 7 and 8. In addition, the material on logic listed tentatively for grades 4-6 is considered standard for grades 7-9. So is:

*57. Experiments on Displacement, Velocity, and Acceleration. These experiments follow P.S.S.C. presentations, with some additions by Professor William Walton of Webster College. Cf. the film Creative Learning Experiences.

F. Junior High Grades; Exploratory Topics.

*58. Introduction of Trigonometric Functions, Via Physical Experiments Using Sand Pendulums, Oscilliscopes, Drawing Boards, etc.

59. Similar Triangles.

*60. Use of Similar Triangles to Find the Height of a Flagpole, etc. (Cf. the film Creative Learning Experiences.)

*61. Graphs of Conic Sections, and Study of Dissectable Plastic Cones.

*62. The What's-It. This is a commercially-available puzzle, actually a cheap ellipsograph. Its study involves similar triangles and ratio and proportion, or else the use of trigonometric functions.

G. Grade Nine.

The Madison Project program for grade nine is discussed in detail in Experimental Course Report # 1, Grade Nine (Report No. 1, June, 1964). Cf. also MLoT.

III. BRIEF HISTORY OF THE MADISON PROJECT

Although we appear to be deferring detailed consideration of "objectives" until rather late in our narrative, there is a reason for doing so. As has been mentioned earlier, "objectives" in any curriculum evolution project must be broad and general at the outset -- only as the work proceeds does it become appropriate to identify narrower objectives. Moreover, as time passes and the effort progresses (meanwhile being paralleled by somewhat similar efforts of other projects), these objectives must necessarily change. Hence, before giving further consideration to objectives, we need to consider a brief history of the Project and to identify some of the shortcomings of the "traditional" curriculum which the Project seeks to overcome.

The project's earliest exploratory classroom work was done in the academic year 1957-8, at the Madison School in Syracuse, New York, with low-I.Q. culturally-deprived 7th graders, under the direction of Professor Robert Davis of Syracuse University, and Mrs. Jane Downing and Mr. William Bowin of the Syracuse Public Schools. These children had, in many cases, not yet learned elementary school arithmetic. Nonetheless, it seemed inadvisable for many reasons to teach them remedial arithmetic: for example, they already disliked arithmetic, there was no reason to suppose that repetition of a teaching procedure that had failed in the past would lead to success in the future, these children badly needed success experiences and a feeling of vitality and challenge in their school work, and arithmetic is, in any case and for any child, a rather isolated small piece of the wide world of mathematics. Hence, the children were taught portions of algebra and analytic geometry. Their response was favorable beyond expectation: they learned arithmetic (which was, of course, repeatedly required in the algebra and geometry), they learned algebra and geometry, they acquired a new enthusiasm for school (for example, truancy decreased markedly), and they seemed to have modified their personal self-concepts and aspiration levels. The instructional procedure was to use "individualized instruction" -- students either worked alone, or else in small groups, and the teachers divided their attention among the groups and individual students. At no time did the teachers attempt to address the entire class as a total group.

This year might be described as an attempt by Project personnel to convince themselves that significant improvement in the curriculum was possible.

1958-9. Because of the preceding year's success with low-I.Q. culturally-deprived seventh graders, the Project sought to explore the potential for curriculum improvement with normal seventh-graders, and with normal and bright fourth graders. Exploratory teaching during 1958-9 focussed on these groups, in middle class neighborhoods. Unfortunately, at this point the Project's previous use of individualized instruction was lost, and was replaced by teaching via working with the entire class as a total group. The reason for this was primarily the involvement of new teachers, not accustomed to individualized or small-group instruction. The procedure of working with the total class as a single group remained normal Project practice until 1963, when various Project advisors (especially Leonard Sealey of Leicestershire, England) reopened this issue. The year 1958-9 began the Project's concern for elementary school children, and for normal (or even bright) children.

1959-1962. In 1959 the geographical focus of Project work shifted from upstate New York to Connecticut, and Mrs. Beryl S. Cochran of the Weston Connecticut Public Schools became the dominant influence on the Project's exploratory teaching. The Project's efforts acquired, thereby, some new emphases, which characterixed most Project work for the next few years:

i) Teaching to the entire class as a single group;

ii) Preference for homogeneous grouping within a school;

iii) Emphasis on working with top groups, in grades 2 through 6;

iv) Following the same children for as many years as possible (usually about5 years);

v) Use of visiting specialist teacher for mathematics, to work with the regular classroom teacher;

vi) Emphasis on demonstrating that bright children in grades 2 through 6 could learn a very large amount of sophisticated mathematics (even completing large sections of high school and college-level mathematics);

vii) Extensive study of how these children learned so much, with particular emphasis on creativity and discovery.[1]

The project's exploratory teaching of this period involved the most sophisticated mathematical content, and the greatest emphasis on creativity, that the Project had ever experienced, and--for grades 3 through 7-- these lessons are probably still the most "advanced" (in these senses)that have been taught by any project thus far.The lesson learned here--and recorded on film, audio tape, and in other ways--that bright children in grades 2 through 7 can learn a very large amount of mathematics in a creative way, and enjoy doing it (in fact, enjoy it immensely), ought not to be forgotten. The Madison Project has subsequently de-emphasized this aspect, because of the very great difficulty of locating teachers who can teach such classes, but the fact remains: much of the mathematics nowadays learned in high school and in college can be learned easily by bright elementary school children in grades 2 through 6, and the children can enjoy it very much. At some time our society must pay heed to this unused potential--particularly in view of the fact that such mathematical experiences are viewed by the children as exciting and pleasurable, not as unpleasant.[1]

---

[1] Cf. the preceding US Office Report, dated 1963, pp. 38 and elsewhere. The quotations from Alfred North Whitehead and E. Paul Torrance can stand as excellent descriptions of the Madison Project's classroom lessons at this time. Fortunately many of these lessons are recorded on film, and a very large number are recorded on audio tape.

These "advanced" classes (especially at Weston, Connecticut) have been studied by psychoanalytically-trained psychiatrists (C. Brooks Fry, M.D., and Carol Fry, M.D.), who have emphasized the children's great eagerness and their unusual absence of anxiety; they have been studied from the point of view of the children's own perceptions and preferences by the clinical psychologist Herbert Barrett; and they have been studied from the point of view of objective measures of mathematical achievement by J. Robert Cleary, of Educational Testing Service.[2] But perhaps most of all the growth of these children over a period of 5 years has been recorded in detail on film, video tape, and audio tape. One can observe how these children approach problems in mathematics and mathematical physics. Their power and their enthusiasm are impressive.

For these "advanced" classes, it is worth adding that "bright" children means approximately the top third of the entire student population in suburban communities such as Weston, Connecticut; the over-all average I.Q. for these "advanced" classes is about 120, but this point should not be pursued in detail since I.Q. does not seem to play an especially decisive role in the achievement of these students. It should not be over-emphasized.

It *is* worth emphasizing, however, that attempts to replicate these results in other schools have not always succeeded. There is apparently some combination of attributes in certain schools that makes it possible to succeed in these schools in a way that is not possible in general. The Madison Project has not succeeded in identifying the attributes of a school that make such "advanced" classes possible, and this is probably one of the Project's most regrettable failures.

Project personnel and consultants *believe* that the attributes necessary for success probably include:

i) general parental and teacher agreement on the desirability of these goals, and on over-all educational goals in general;

ii) creative, flexible, well-educated teachers;

iii) a very flexible school administration and school organization, that supports both teacher and student in every possible way;

iv) a non-authoritarian atmosphere, in the class, in the school, in the community, and in most of the homes;

---

[1] Cf. *Goals for School Mathematics*.(B 1963), and also Davis (A 1964i).
and

[2] Cf. Cleary (A 1965).

v) respect for children as people;

vi) general satisfaction of children's needs, including needs for physical activity, affection, attention, autonomy, etc.;

vii) mutual respect, and even affection, among all the people involved; parents, teachers, children, and administrators;

viii) a general atmosphere of maturity, flexibility, creativity, and cooperation in the school and in the community.

The identification of necessary attributes is elusive. Nonetheless, the fact remains, in some schools, but not in others, it is possible for "bright" elementary school children to learn a large amount of high school and college mathematics, in a creative way, and enjoy doing it.

No one who witnessed the Weston classes can escape from the feeling that this fact must be important.

1961-present. In 1961 the Project opened a third office, at Webster College, in Webster Groves, Missouri.

(At the present time the Project has three offices: at Syracuse University, Syracuse, New York; at Weston, Connecticut; and at Webster College.) Since 1961 the Project has cooperated in a variety of teacher training programs at Webster College, to the extent that now emphasis have appeared in Project work:

i) The major emphasis has been on teacher education, both pre-service (college undergraduate) and in-service.

ii) In terms of student selection, the Project began in 1961 working primarily with "ordinary" college-bound students, in "ordinary" school situations, but still with considerable emphasis (wherever possible) on the use of specialist mathematics teachers for the intermediate grades.[1]

---

[1] However, it should be added that the years 1964 and 1965 have shown a swing away from the use of "visiting specialist teachers" for mathematics in elementary schools, and a greater emphasis on helping the classroom teacher to do the job satisfactorily. "Specialist teachers" remain just as important, but they work mainly with the teachers, and not with students.

iii) The Project began to be concerned with larger numbers of teachers and larger numbers of students, spread geographically over much of the United States (and even in Canada, Australia, England, and Africa).

iv) As a result of the preceding emphases, the Project began to distinguish more sharply between tentative, exploratory lesson sequences, and reliably-tested stable lesson sequences. Only the latter were used in large-scale teacher education work.

v) The Project had a number of unsatisfactory experiences in attempting to work with junior high school students (grades 7 and 8). Students at this grade level pose especially difficult problems for curriculum planning; this has been, in general, the experience of a large number of curriculum projects.[1] Probably much more study of children this age is needed, together with a far greater effort to use the results of such studies in designing school programs, and in allocating grades among the various buildings (e.g. 8-4, or 6-6, or 4-4-4, or 6-3-3, etc.).

1963-present. Beginning in 1963, several new emphases appeared:

i) The students whom the Project follows for 3, 4, or 5 consecutive years were, of course, getting older and moving into later grades. Largely as a result of this, together with unresolved doubts about the program for grades 7 and 8, the Project began to focus considerable effort on grade 9.[2]

ii) At the other end of the age scale, the Project reviewed its exploratory teaching in nursery school, kindergarten, and grades 1 and 2 -- which had until then been desultory and disorganized -- and began a concerted systematic attack on exploratory teaching of mathematics at this grade level.[3]

Perhaps most important are the following returns to earlier directions:

iii) The Project began large-scale work with culturally-deprived urban children, in St. Louis and in Chicago, with unexpectedly gratifying results.[4]

---

[1] Cf. Davis (A1964c), also Murphy (B1965)

[2] Cf. Davis (A1964g)

[3] Cf. Doris Diamont Machtinger, Experimental Course Report/Kindergarten. (Report No. 2, July, 1965), the Madison Project (in preparation).

[4] Cf. The films Postman Stories, Guessing Functions, and Graphs and Truth Sets.

iv) The Project, stimulated in part by Leonard Sealey of Leicestershire, England, has renewed its original interest in individualized instruction and in the procedure of dividing a class of (say) 30 students up into "committees" or "small groups" of about 3 or 4 each. In either case, the teacher seldom if ever stands at the front of the room and addresses the entire class. Instead the teacher sits with one group for awhile, then moves on to another, etc.[1]

Finally, as a result of the desire to let students work alone or in small groups, of the desire to combine some mathematics with some physical science, and of the desire to reach "non-verbal" children whose ability may be great but who do not function naturally in a world of verbal behavior and abstractions, the Project has acquired another emphasis:

v) The Project has given new impetus to its efforts to produce "individualized study materials," in the form of a library of "shoe-box" kits for various scientific experiments, mathematical puzzles, etc.

---

[1] Cf. the film I Do - And I Understand, made by the Mathematics Teaching Project, Nuffield Foundation, St. Dunstan's College, London, England.

IV. OBJECTIVES

The "objectives" of a curriculum evolution project include:

1. Objectives for student learning and student growth;
2. Objectives for teacher growth;
3. Objectives for the evolution of the curriculum;
4. Objectives for curriculum theory;
5. Objectives for the operation of the project itself.

What this list is intended to suggest might be made clearer by some examples:

The meaning of "student learning" or "student growth" is probably clear, in general, since schools have always had objectives for student growth (for example, that students in the primary grades learn to count pebbles accurately, or that older students come to understand the concept of *variable*).

Items 2 and 3 may be less clear, since they are less commonly encountered in school situations. "Teacher growth" refers to the important matter of how much the *teacher* learned this year, and how much he grew -- in terms of understanding the subject matter more deeply, being able to work with it more creatively, and gaining power as an observer of children and as a person who influences the growth and learning of children. "Evolution of the curriculum" may seem meaningless, since in recent years the curriculum has been largely frozen and has *not* evolved; however, it is assumed that curriculum evolution should be the normal and expected thing in any school. Last year's fourth grade program should not also serve as next year's fourth grade program, except perhaps briefly and occasionally while curriculum effort focuses elsewhere for a time.

Schools need to provide a genuine allocation of resources in support of "teacher growth" and "curriculum evolution," and some schools are beginning to do so to an increasingly significant extent. We shall return to this question in Section V.

Finally, a curriculum project should be conducted, if possible, in such a way that it can make contributions to the accumulation of general curriculum theory, and it must have objectives (such as reasonable efficiency) that apply to the behavior of the project itself.

In the case of the Madison Project, these objectives can be stated as follows:

A$_1$. Objectives for Student Learning and Student Growth — "Cognitive" or "Intellectual" Objectives.[1]

---

[1] Cf. Davis,
I. (A1963b)
II. (A1964e)
iii. *Matrices, Logic, and other Topics*, Addison-Wesley Publishing Co., Reading, Massachusetts. (In preparation.)

1. We want the child to develop his ability to discover patterns in abstract situations;

2. We want the child to develop the kind of independent exploratory behavior that goes beyond anything the teacher has suggested, to try out new possibilities or make further observations;

3. We want the child to acquire an appropriate set of mental symbols that represent mathematical situations in a pseudo-geometric pseudo-isomorphic fashion, in the sense of Polya and Leibnitz;

4. We want the child to recognize fundamental mathematical concepts, such as *variable*, *function*, *graph*, *matrix*, *isomorphism*, etc., to be able to work with these concepts, and to be able to build future learnings on these concepts as a foundation;

5. We want the child to acquire a reasonable degree of mastery of fundamental mathematical techniques;

6. We want the child to know basic mathematical facts -- for example, the fact that ${}^-1 \times {}^-1 = {}^+1$.

A_2. Objectives for Student Learning and Student Growth -- "General" or "Attitude" Objectives.

7. We want the child to come to believe that mathematics really is discoverable;

8. We want the child to arrive at a reasonably realistic assessment of his own individual ability to discover mathematics;

9. We want the child to come to recognize the open-endedness of mathematics;

10. We want the child to develop an honest self-critical ability;

11. We want the child to come to value abstract rational analytical thinking;

12. We want the child to recognize the value of "educated intuition";

13. We want the child to come to regard mathematics as "fun" or "exciting" or "challenging" or "rewarding" or "worthwhile";

14. We want the child (and, especially, the older child) to gain some perspective on the heritage of accumulated culture, particularly as the history of ideas involves mathematics, in relation to broad historical perspectives, from pre-historic times until the present;

15. We want the child to be at home in using mathematics in physical or social problems.

Perhaps objectives 14 and 15 should be considered cognitive, rather than attitudinal. However, objectives 7 through 13 are not primarily "verbal" or superficially intellectual. Objective 7, for example, would not be identified by the fact that a student says that mathematics is discoverable; it could only be inferred from his approach when confronted with certain new mathematical situations: does he try things out, does he modify the problem, does he attack it tentatively and explorationally from various angles -- or does he give up and wait for a suggestion on how to proceed?

Similarly, objective 8 would be judged from those situations in which the student tries to discover an answer, how long he perseveres, etc. Obviously, good judgment on one's own ability to discover mathematics can come only from considerable experience and with reasonable maturity.

Again, for objective 9 one would ask whether the child attempts to extend mathematics further or not, and whether he is receptive to other students' attempts to make extensions. A very fine instance of what is meant here can be observed, in an actual classroom lesson, in the film entitled Graphing an Ellipse (with seventh-grade children who are at that time in their third year of working with Madison Project materials); a brief excerpt of this lesson, including the relevant "extensions beyond the frontier" is contained in the film entitled Education Report: The New Math. A separate instance can be observed in the film entitled Complex Numbers via Matrices, the later (and relevant) portion of which is also separately available under the title Report to the Miami Meeting; a third, and more extended, instance can be viewed in the film Matrices (with students in grades 5 and 6).

For objective 10, again, one would observe actual student behavior, in at least three situations:

i) When the student makes an error, how often does he discover the error himself before anyone else suggests it?

ii) When the teacher, or another student, points out an error, does the student who made the error try to understand what was wrong, or does he attempt to defend his original (and erroneous) response?

iii) When a student is in fact correct, but is confronted by the massive opposition of his classmates, does he retain faith in his original logic, or does he allow himself to be overwhelmed by the (erroneous) majority opinion?

Many instances of relevant behavior can be observed in various films, for example in the films Graphing an Ellipse and Complex Numbers via Matrices.

In a good school situation, teachers, too, must learn and grow each year. Indeed, this is one further reason for introducing algebra and geometry into the elementary school curriculum, since it opens up to the teacher a new opportunity for understanding the mathematical foundation upon which arithmetic is based. Arithmetic by itself never fully "makes sense." It begins to do so much more satisfactorily when it is taken in its proper algebraic and geometric context. If the primary advantage here lies with the student, there is an important, and quite parallel, advantage to the teacher.

B. Objectives for Teacher Learning and Teacher Growth.

1. With each additional year of teaching, the teacher should feel that her mutual adventures in exploration and discovery, together with her students, have allowed her to feel more at home with a broader range of mathematical topics.

2. With each additional year of working with children, and thinking reflectively about what she is doing, the teacher should become a better observer of child behavior (including learning and growth), and a more profound interpreter of her observations.

3. With each additional year of working with children, the teacher should become more skillful in guiding student learning and student explorations.

4. A really good school situation should also make provision for teachers to learn from one another, and from specialist teachers, and to teach one another. An effective program of this sort can make it far easier to cope with the demands of a continually-evolving curriculum.

C. Objectives for the Evolution of the School Curriculum. [1]

1. The curriculum needs to be gradually broadened to include topics presently omitted, such as statistics, logic, matrix algebra, digital computers, etc.

The present "traditional" curriculum was devised for a world of retail sales, surveying, navigation, and agricultural mathematics. The world has changed, and the importance of various mathematical topics has changed accordingly, both in response to pressures from external society, and to pressures arising from an internal reorganization within mathematics itself.

---

[1] Cf. Davis,
i. (A1963$_a$)
ii. (A1964^l)
iii. (A1963^e)
iv. *Matrices, Logic, and other Topics*, Addison-Wesley Publishing Co., Reading, Massachusetts. (In preparation.)
v. (B1965)

2. There needs to be more diversity of types of activity in the student's day in school. Mathematics is often a "reading, writing, and reciting" subject, a "paper and pencil and textbook" subject, which many children, especially around grades 7 and 8, find unappetizing and unpleasant. Mathematics need not be taught in this way: it can involve going outdoors for surveying, for finding the height of the school flagpole, and for velocity and acceleration studies on automobiles; it can involve actually working with a "real live" digital computer; it can involve engineering drawing with drawing boards; it can involve small-group or "committee" study of individualized instruction materials; it can involve laboratory work with physical apparatus; it can involve the analysis of mathematical puzzles, or the playing of mathematical games that are carefully chosen so as to be closely relevant to basic mathematical concepts; and it can involve lively seminar discussions as a group of students seek to "invent" their way out of a seeming mathematical cul-de-sac, or to explore a new kind of mathematical situation.

3. In fact, there needs to be a more active role for the student, and one in which he has enough decision-making autonomy to be able to learn some meaningful lessons.

4. The pace of learning can easily be speeded up, and needs to be. At present it is too slow. Speeding it up need not make it harder, even for less gifted students, provided the speed is achieved by eliminating that which was unnecessary (or often downright confusing), and by handling that which remains with good pedagogical and mathematical effectiveness. A very slow pace does not make a subject easier -- but it does give less gratification to the learner.

5. In general, the vitality and relevance of the school program must be increased, and along with them there needs to be a deeper sense of student involvement. It has been remarked of the "traditional" program, which is often dull and unrewarding, that many school dropouts never dropped out -- they couldn't, because they never dropped in.

6. Within the presentation of mathematics, itself, improvements in definitions, notation, sequencing of topics, etc., need to be made.

7. On present evidence, some primary grade children can handle sophisticated abstractions far beyond any they would ordinarily encounter in a traditional program; probably this ability should be exercised, developed, and utilized.

8. On the opposite side, many older children -- and especially 6th, 7th, and 8th graders -- need more opportunities to work with physical apparatus of one sort or another.

If the preceding objectives are (mostly) strategic objectives or "goals," the following might be listed as tentatively held tactical objectives, or possible means for achieving some of these ends:

9. The new curriculum developments may profitably be based upon carefully-devised and carefully-tested lessons, or sequences of lessons, that are rather fully structured (although they should usually be flexible enough to provide considerable student autonomy). These "carefully-devised lessons, or sequences of lessons" are what we have earlier labelled creative learning experiences.

This approach is quite different from leaving it up to the teacher to devise her own lesson plans for every lesson for every day. Creative learning experiences do not minimize the role of the teacher -- by no means! -- but the use of such previously-worked-out lessons shifts the teacher's role, primarily by giving her more potent tools to work with.

10. More use of individualized and small group instruction should help the teacher to assume a role that is more effective than "lecturing" to the total class of thirty or forty children at one time.

D. Objectives for Curriculum Theory.

1. What is done by curriculum projects should become the subject of a dialogue involving the academic community generally, and especially those professionals who work in curriculum, or who are concerned with teaching methods and the broad areas of instruction. For example, projects should relate closely to organizations such as the Association for Supervision and Curriculum Development (ASCD), and other relevant organizations within the NEA and elsewhere.

2. While over-elaborated "systems" are seldom, if ever, the answer to educational needs, nonetheless a reasonable collection of the wisdom acquired by curriculum-evolution projects should be accumulated within the relevant professional literature.

E. Objectives for the Operation of the Madison Project Itself.

These are obvious, such as:

1. The Project should achieve its goals as efficiently as possible.

2. The Project should make every reasonable effort to determine the appropriateness of the changes which it attempts to produce.

3. The Project should achieve the involvement and active participation of as large a part of the relevant academic and professional community as possible.

4. The Project should respect the different viewpoints of others, and exercise restraint in pursuing its own goals and in presenting its own recommendations.

5. The Project should attempt to remain flexible, and even self-renewing; it should not become wedded to last year's point of view.

Obviously, these objectives are in some measure contradictory, and the day-to-day operation of a curriculum evolution project involves many compromises in the face of irreconcilable goals.

V. FURTHER DISCUSSION OF PROCEDURE

For the most part, Madison Project procedures have been discussed in previous reports (cf., for example, the preceeding USOE report, dated 1963), and these discussions will not be repeated here. Two new aspects, however, deserve consideration: the first is provision for teacher education and curriculum evolution within the school situation and provided by the board of education; the second is curriculum work in large cities.

1. Provision for Teacher Education and Curriculum Revision Within the School Situation.

Concerning the first part of this matter, teacher education, a study by the sociologist David Street, of the University of Chicago, concluded that the police department in most communities spends more of its budget on education than the board of education does -- if one refers to the education of the organization's own operating personnel. The police department typically maintains rifle ranges, weapons instructors, physical combat instructors, instruction in matters of law, etc., for the sake of the department's own operating personnel. The board of education usually fails to match this record. Yet, if curriculum evolution is ever to become a fact, instead of a rather marginal "frill," continuing teacher study must also become part of a school system's standard operating procedure.

Concerning the second part of this matter, one prominent school administrator, Dr. John Sternig of Glencoe, Illinois, in a recent presentation at an ASCD meeting in Chicago emphasized that "the teacher's job is curriculum and instruction." By keeping the teacher standing in front of a classroom too many hours per week we have made it impossible for him to do his job -- that is to say, we have made it impossible for him to do the curriculum part of his job, without which the instructional part cannot be done as well as it should be.

Fortunately, the dim outlines (at least) of a method by which schools can begin to make more adequate provsion for teacher education and curriculum are now beginning to appear. Much of the theoretical argument has occurred under the leadership of Professor Jerrold Zacharias and the President's Committee on Research ond Development in Education, of which Professor Zacharias is chairman.

Briefly, the suggested scheme would identify several "layers" or "levels" of teachers within a school system. The classroom teacher might be designated as a "T_1-level teacher." Some classroom teachers would be selected (or, perhaps would select themselves) for special study in specific areas (for example, in mathematics and in the curriculum evolution work in mathematics), and would assume responsibility for serving either as a resource person to their colleagues, or as a "visiting specialist teacher" in (say) mathematics, or as a teacher of their colleagues (who would conduct in-service workshops or seminars for T_1-level teachers). These specialist teachers we shall designate as "T_2-level teachers." Where necessary, one might go further: some T_2-level teachers would be selected for additional study, and would assume some kind of "resource" or "advisary" responsibilities in relation to T_2-level teachers. Teachers fulfilling this role might be designated as "T_3-level teachers."

The task of curriculum evolution, in any given school system, would then focus at first on T_3-level teachers, whose responsibility would be to work with T_2-level teachers. The T_2-level teachers would work with the classroom teachers.

In actual fact, past experience with "helping teachers" or "curriculum specialist teachers" in large cities suggests that T_3-level may be going too far; these T_3-level teachers may become too far removed from the local classroom level to be effective. T_2-level, however, can be made to work, as will be seen presently -- especially in view of the fact that the T_2-level teacher has a dual responsibility: he teaches other teachers, *but he also teaches children in a regular classroom situation*.

The arithmetic of such an organization is enlightening: in the Chicago Public Schools, for example, there are 18,000 teachers involved in mathematics in grades K-8. If 600 of these are selected for special study, and become T_2-level teachers, then 600 T_2-level teachers must attempt to reach 18,000 T_1-level teachers. That is to say, each T_2-level teacher must attempt to work with 30 classroom (or "T_1-level") teachers. These numbers are reasonable and realistically possible.

Thus, the 18,000 teachers in Chicago do *not*, in fact, pose an insuperable obstacle to curriculum evolution. Indeed, if a continuing operation of T_2-level teachers can be achieved, even the rapid personnel turn-over of a large city does not constitute an insuperable obstacle to an on-going program of curriculum evolution.

Evidently there are problems that must be solved before an effective structure of T_2-level teachers can be achieved. Identifying T_2-level teachers is not difficult, as we shall see presently, and even providing the appropriate education for T_2-level teachers is not too hard to arrange. But what happens next? How shall T_2-level teachers be paid for the added effort and responsibility they have been asked to assume? How can their new role as "teachers of teachers" best be reconciled with their continuing role as classroom teachers? *When* shall the in-service workshops conducted by T_2 teachers, and attended by T_1 teachers, occur? After school? During school? On Saturdays? What recognition, if any, shall the T_1-level teacher receive for attending these workshops? Will there be a tendency to develop an "in-group" of T_2-level teachers who may soon become a conservative force, rather than a progressive one? What can be done to avoid this?

If schools are to allocate a significant fraction of their budget to curriculum evolution and continuing teacher education, the "T_1, T_2 level" organization may provide part of the structural framework for this part of the school program.

We turn now to some practical attempts to make such a scheme work.

2. Curriculum Evolution in Urban School Systems.

The use of the arithmetic of the Chicago Public Schools in the preceding section was not accidental, for one of the earliest instances of Madison Project involvement in urban curriculum work did in fact occur in Chicago.

An informal use of "T_2-level teachers" is not a new idea in private schools and in suburban schools. Indeed, under the present USOE "demonstration project," the Madison Project has had specialist teachers at work in Clayton, Missouri; Lincoln, Massachusetts; Weston, Connecticut; Greece, New York; and elsewhere, and these specialist teachers have played essentially a "T_2-level" role. Moreover, they have been doing so for several years, and consequently they now constitute a definite reservoir of professional talent able to play a new and essential role, as we shall presently see.

The point at which the "T_1-T_2-level" organization creates important new possibilities is in school systems of our largest cities. Madison Project involvement in very large scale urban problems began in St. Louis, under the administrative leadership of Dr. Samuel Shepard, Assistant Superintendent of the St. Louis Public Schools, and in Chicago, under the administrative leadership of Mrs, Evelyn Carlson, Associate Superintendent in Charge of Curriculum. Subsequent efforts have occurred in San Diego County, California, under the administrative leadership of Dr.John Huffman and Dr. Jack Price of the San Diego County Board of Education, in New York City, under the leadership of Deputy Superintendent Dr. Joseph O. Loretan, and in Washington, D.C. as part of the "model school district recently created there. Three distinct patterns of work are involved, as will be illustrated by the examples of Chicago, St. Louis and Washington, D.C.[1]

The Case of Chicago. Before turning to actual operational details, it is worth considering the view from the Curriculum Director's office in any of our largest cities. In order to be able to make and to implement significant curriculum decisions, the Curriculum Director must be able to arrange for appropriate teacher education, having (among others) these three attributes:

i) It must be on a large enough scale to be meaningful in today's urban terms: where there are 18,000 teachers to reach, 600 may be a significantly large group, but 35 probably is not;

ii) It must operate quickly enough to be meaningful; a program that requires 3 years is reasonable, and one that requires 5 years may be. A program that requires 15 years, or 30 years, is not realistic. At that rate, the world is changing faster than the curriculum, and the schools are moving backward in relation to outside society;

iii) The teacher education program must be matched to the city's curriculum plans.

Programs of the type described above have not existed in the past in urban school curriculum work. A hypothetical example may make this clearer. Suppose that, after suitable consultation, discussion, and planning, a Curriculum Director in a large city decided to introduce spoken Russian as a subject of study available to all 5th grade children. To do this, he requires at least 300, and perhaps two or three times that many, experienced 5th grade teachers who can speak Russian and teach it effectively. Under traditional arrangements this is probably unattainable: colleges and universities in the city, usually jealous of their autonomy, will probably offer programs (if they offer any) that can cope with 25, or perhaps 100 students, that will not reach the required level of proficiency within a realistic time, and that will probably not relate their instruction to the actual curriculum plans of the city schools.

---

[1] San Diego County and New York City parallel reasonably closely the arrangements described for Chicago.

Yet the examples of the United States armed forces, and of the Peace Corps, show that effective education of this sort, reaching adequate levels of proficiency within a short time, is possible.

To do his job satisfactorily, it is evident that the big city Curriculum Director must be able to command educational resources of this type, as the armed forces and the Peace Corps do at present.

One of the first curriculum directors to create such a program was Mrs. Evelyn Carlson, Associate Superintendent in Charge of Curriculum, of the Chicago Public Schools. The Chicago Bureau of Curriculum spent the academic year 1963-4 in making plans, with the cooperation of a panel of outside consultants on mathematics, working together with the Bureau's own mathematics staff, headed by Mrs. Bernice Antoine. By the summer of 1964 plans had been made, and in August, 1964, a two-week workshop for prospective T_2-level teachers was conducted. During the spring of 1964, prospective T_2-level teachers had been identified by principals and district superintendents, and those chosen were invited to attend the workshop. Teachers in grades K-3 attended sessions at Chicago Teachers College North, under the direction of Professor Patrick Suppes of Stanford University. Teachers of grades 4-8 attended sessions at the Byrd School, under the direction of Professor Robert B. Davis and a staff of 41 colleagues from the Madison Project. Since Madison Project involvement was limited to the sessions at the Byrd School, this is all that need be described here.

The Workshop at Byrd School, August, 1964. Byrd School is an elementay public school, equipped with two television studios, and with closed-circuit TV available in every classroom. The arrangements easily accommodate 600 teacher-participants, and in fact approximately 400 attended. In each classroom there were one or more of the 41 Madison Project teachers, who had for several years been fulfilling a T_2-level role in suburban schools, private schools, or in teacher-training programs in colleges and universities. Those 41 teachers nearly all had both experience teaching sophisticated mathematics at high school or college levels, and also experience teaching Madison Project methods and materials to elementary school children and to elementary school teachers. For "large group" instruction, it was possible to teach all 400 teachers at once via the closed circuit TV.

The impact of the Byrd workshop depended upon three factors:

i) The existence of the 41 experienced "T_2-level" teachers, accumulated from the Madison Project's continuing cooperation with suburban schools, private schools, and colleges, under the present USOE demonstration project.

ii) The existence of 400 elementary teachers in Chicago with the potential ability to assume a "T_2-level" role as a "teacher-of-teachers," and as an active agent in curriculum evolution work.

iii) The existence of a repertoire of fully-worked-out classroom lessons (the so-called "creative learning experiences" discussed earlier), which made possible an "inverse" approach to teacher education. Instead of the teachers learning new mathematical ideas and being left on their own to devise ways to help children learn these ideas, the starting point for the teachers' own study was consideration of an actual classroom lesson (presented either via film or via closed-circuit TV), after which the teachers considered the mathematical points involved in the lesson.

Remark 1. The problems of the continuing growth of teachers has long plagued educational structure. Teachers all too often reach their peaks all too early in their professional careers, or else "advance" by leaving their classroom and becoming guidance counsellors, administrators, or college teachers. The "T_1, T_2" organization may serve to provide growth opportunities for teachers without requiring them to abandon their classroom.

Remark 2. It may seem that the use of "fully-worked-out" classroom lessons produces inflexibility, and erodes the role of the teacher. Examination of the Madison Project's "fully-worked-out" lessons should make it clear that the teacher's role is not less important in this kind of teaching, nor is the lesson format inflexible. The lessons are designed to be student-centered and to vary in response to student needs and student initiatives. "Heeding the student" is not the same thing as "making unnecessary errors."

Follow-up: the academic year 1964-5. During the academic year 1964-5, two one-term courses were offered through Chicago Teachers College North and Chicago Teachers College South with the collaboration of Madison Project specialist teachers. The fall course was a follow-up for participants from the August workshop, whereas the spring course was a beginning course for a new group of prospective T_2-level teachers. In addition, Madison Project specialist teachers visited schools in Chicago, observed classes, taught demonstration classes, conducted in-service workshop sessions, and participated in planning sessions.

The Byrd Workshop, August, 1965. In August, 1965, a second two-week workshop was conducted, generally parallel to that of 1964, but primarily concerned with continuing the education of Chicago teachers who had attended one or more of the earlier sessions.

A few additional "new" teachers were allowed to enter the program, also, at this time. By the end of the 1965 August workshop, it was possible to identify a definite group of Chicago teachers who appear capable of assuming the T_2-level role successfully.

September, 1965, and after. The Chicago program is presently planned to continue and to grow larger, as well as to be more officially structured within the pattern of overall school operations. At the present time financial support for a continuation and expansion of this program is being sought from the Cooperative College-School Science Program of the National Science Foundation. Costs thus far have been borne mainly by the Chicago Board of Education, with some contributions form the Madison Project as part of the USOE demonstration project.

The distinctive features of the Chicago operations are perhaps these:

i) It is city-wide, and attempts to reach equally every child in the Chicago Public schools;

ii) The central role in the education of T_2-level teachers is assigned to the very intensive two-week summer workshop conducted each summer, and taught by 40 or so experienced Madison Project teachers from all over the United States;

iii) The T_2-level teachers themselves have the crucial assignment of reaching all (or as near to this as possible) of the remaining teachers in grades K through 8 for the entire city of Chicago -- that is to say, 18,000 teachers.

St. Louis. The method of operation in St. Louis is quite different from that in Chicago, and takes advantage of the fact that the main office of the Madison Project is located in St. Louis, as is Webster College. In September, 1963, under the administrative leadership of Dr. Samuel Shepard, Mr. Knowles Dougherty, a Madison Project specialist teacher, began a full-time assignment working with teachers of the Banneker District elementary schools. The teacher education program was based on the Madison Project's In-Service Course I, a "packaged" in-service course that combines films showing actual classroom lessons, with printed materials that explain both mathematical and pedagogical aspects of the lessons. It is significant that Dr. Shepard arranged for this activity to occur in the schools during the regular school day. With this as a beginning, those St. Louis teachers who chose to continue their education further in this direction enrolled, at their own expense, in the Ford Foundation-supported MAT program at Webster College, created under the administrative leadership of Sister M. Jacqueline Grennan, S.L., then Vice-President of Webster. This MAT program was created pedagogically and mathematically by Professor Katharine Kharas, by Donald Cohen, and by Mr. Dougherty, and is based on the methods and materials of the Madison Project. The MAT program is intended for experienced superior elementary school teachers who wish to become mathematics specialists; it involves the study of 30 credits of mathematics, making use (among other things) of the Madison Project's "creative learning experiences," thereby insuring relevance to a modern school mathematics program.

Mr. Dougherty's work in the Banneker District was continued in 1964-5; a somewhat similar but slightly modified program began in September, 1964, in the Long School District, under the adminsitrative leadership of Lucille Sutherland and Kathleen A. Wall, and was conducted by Louis Cohen, a Madison Project specialist teacher. At the present time, the work in the Banneker district is conducted by Mr. Ogie Wilkerson, of the St. Louis Public Schools.

Evidently, the method of operation in St. Louis takes longer, involves smaller numbers of teachers, and--thanks to the 30 credit MAT program at Webster College--penetrates to deeper levels than the program in Chicago. It should, however, be strongly emphasized that the Webster College MAT program is not just "any" college graduate study: it is designed specifically to enable elementary teachers to become mathematics specialists, it is taught by Madison Project specialist teachers, it involves 30 credits of mathematics, and it utilizes the methods and materials of the Madison Project. Thus it does in fact continue the education which the Madison Project's In-Service Course I begins.

Washington, D.C. The method of operation in Washington, D.C. has been entirely different from either the Chicago or the St. Louis patterns, In the early months of 1965 a "special school district" was designated within the Washington, D.C. Public Schools consisting of Cardoza High School, the 4 junior high schools which send students to Cardoza, and the 14 elementary schools that send students to these three junior highs. These schools were to operate as a separate unit within the Washington system and to attempt to become nearly ideal examples of the best present practice in instruction and curriculum, with special emphasis on modern curriculum evolution efforts.

During the summer of 1965, a seven-week program for both students and teachers from these schools was operated by the Model School District. The involvement of the Madison Project, under the administrative leadershop of Mrs. Emma Lewis and Miss Gail Saliterman, involved the teaching of a 6 week course for teachers by Louis Cohen, and the teaching of a 6-week demonstration class of children by Miss Mary Jane Hopkins. Both Mr. Cohen and Miss Hopkins are Madison Project specialist teachers. Follow-up action, if any, has not yet been determined.

Obviously, the Washington pattern can have the advantage of very effective K-12 coordination, at the price of working with a relatively small number of schools, teachers, and children.

VI. RELATED RESEARCH AND CURRICULUM EFFORTS

The Madison Project's "creative learning experiences" are devised by Project teachers working in actual classroom situations, and are more or less new and original. They do, nonetheless, owe debts to various sources, some of which are identified in this section.

A. Mathematical Ideas.

To separate "mathematical ideas" from "pedagogical ideas" is not entirely possible, but to the extent that such a separation can be made, the Project owes debts for "mathematical ideas" to the following published works:

Carl B. Allendoerfer and Cletus O. Oakley, Principles of Mathematics, McGraw-Hill, New York, 1963.

Max Beberman, An Emerging Program of Secondary School Mathematics, Harvard University Press, 1958.

Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra, Macmillan Co., New York, 1953.

Howard Eves, Introduction to the History of Mathematics (revised edition), Holt, Rinehart, and Winston, New York, 1964.

Eves and Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics, Holt, Rinehart, and Winston, New York, 1964.

Robert M. Exner and Myron F. Rosskopf, Logic of Elementary Mathematics, McGraw-Hill, New York, 1959.

John G. Kemeny, Laurie J. Snell, and Gerald L. Thompson, Introduction to Finite Mathematics, Prentice-Hall, Inc., 1957.

E.E. Moise, Geometry from an Advanced Standpoint, Addison-Wesley Publishing Co., Inc., Reading, Massachusetts, 1963.

F. Mosteller, R.E.K. Rourke, and G.B. Thomas, Probability: A First Course, Addison-Wesley, 1961.

David Page, Number Lines, Functions, and Fundamental Topics, Macmillan, 1964.

School Mathematics Study Group, First Course in Algebra, Yale University Press, 1960, 1961.

and to unpublished suggestions received from:

Robert Karplus, Physics Dept., Berkeley

David Page, University of Illinois and ESI

Paul Rosenbloom, MINNEMAST and Teachers College, Columbia University

W. W. Sawyer, Wesleyan University

Robert Rosenbaum, Wesleyan University

Andrew Gleason, Harvard University, Mathematics Dept.

Frederick Mosteller, Harvard University, Dept. of Statistics

E. E. Moise, Harvard University, Mathematics Dept.

R. M. Exner, Syracuse University, Mathematics Dept.

Erik Hemmingsen, Syracuse University, Mathematics Dept.

Gerald Thompson, Carnegie Institute of Technology

George Springer, Indiana University

Francis Friedman,† MIT, Physics Dept.

Jerrold Zacharias, MIT, Physics Dept.

Earl Loman, MIT, Physics Dept.

Malcolm Skolnik, MIT, Physics Dept.

David Hawkins, University of Colorado, Dept. of Philosophy

Max Beberman, University of Illinois, UICSM

Morris Kline, NYU

Lipman Bers, NYU

Marshall Stone, University of Chicago

Saunders MacLane, University of Chicago

Edward J. McShane, University of Virginia

Gail Young, Tulane University

Alvin Feldzamen, University of Wisconsin

† Deceased

George Polya, Stanford University

William Walton, Webster College, Physics Dept.

Frederick Cunningham, Bryn Mawr

Raymond Redheffer, UCLA

Al Putnam, University of Chicago

B. Specific Pedagogical Ideas

Specific pedagogical ideas have been received from the works, published and unpublished, of George Polya, W. W. Sawyer, David Page, Max Beberman, Leonard Sealey, Thomas Clayton, Norman Levinson, Phillip Franklin, and Frederick Mosteller.

C. Rationale Concerning Schools and Learning

The broad rationale concerning the nature of schools and the nature of learning has been influenced deeply by:

Pierre Boulle, The Test, Popular Library, New York, 1960.

Ruth Brecher and Edward Brecher, "Gifted Children Need Freedom to Learn," Parent's Magazine, June, 1962, pp. 44 ff.

Nathaniel Cantor, The Dynamics of Learning, Henry Stewart, Inc., New York, 1961.

Z. P. Dienes, The Power of Mathematics, Hutchinson Educational Limited, London, 1964.

Jules Henry, "American Schoolrooms: Learning the Nightmare," Columbia University Forum, Vol. 6, No. 2, Spring, 1963, pp. 24 - 30.

John Holt, How Children Fail, Pittman, 1964.

Earl C. Kelley, Workshop Way of Learning, Harper, New York, 1951.

Kurt Lewin, "Field Theory and Learning," pp. 60 ff, in Kurt Lewin, Field Theory in the Social Sciences (edited by Dorwin Cartwright), Harper Torchbooks, 1951.

Hughes Mearns, Creative Power, Dover Publications, New York, 1958.

A. S. Neill, Summerhill, Hart Publishing Co., New York, 1960.

Theodor Reik, Listening With the Third Ear, Farrar, New York, 1948.

Carl Rogers, Client-Centered Therapy, Houghton Mifflin Co., Boston, 1959.

Joseph J. Schwab, "The Teaching of Science as Enquiry," in Schwab & Brandwein, The Teaching of Science, Harvard University Press, 1964.

E. Paul Torrance, Creativity, Department of Classroom Teachers, American Educational Research Association of the National Education Association Research Pamphlet Series, "What Research Says to the Teacher," #28, April, 1963.

Alfred North Whitehead, Aims of Education, Mentor Books, Macmillan Co., New York, 1929.

among others, by considerable personal contact with Professor Jerome Bruner, Director of the Harvard University Center for Cognitive Studies, Dr. Richard Anderson of the University of Illinois, Professor Gilbert Findlay[†] of the University of Illinois, Professor Donald Snygg of Oswego, Professor Carl Pitts of Webster College, Professor Richard deCharms of Washington University, Professor J. Richard Suchman of the University of Illinois, Dr. C. H. Bagenstose, M. D., Dr. C. Brooks Fry, M. D., Dr. Carol Fry, M. D., Leonard Sealey, Layman Allen, and Professor Warwick Sawyer.

Other influences have included a small but important personal contact with Jean Piaget, the speeches and writing of J. Lloyd Trump, the "dual progress plan" of Dr. George D. Stoddard, and the deeply inspiring example of Nova School, in Ft. Lauderdale, Florida, as well as the recent work on computer-based teaching machines and "responsive environments" by Professor Patrick Suppes and Professor Omar K. Moore.

D. Rationale Concerning Curriculum Evolution and Administrative Procedures.

In the important area of the process of curriculum evolution itself, with attendant administrative questions, the Project has been influenced especially by Professors Jerrold Zacharias, Phillip Morrison, and Douglas McGregor of M.I.T., by Professor David Hawkins of the University of Colorado, by Max Beberman, by Professor Donald E. Kibbey of Syracuse University, by Sister M. Jacqueline Grennan, S.L., of Webster College, by Mrs. Evelyn Carlson of the Chicago Public Schools, and by Dr. Samuel Shepard of the St. Louis Public Schools.

[†] Deceased

VII. RELATED PROGRAMS

The USOE demonstration project is focused on working with schools to achieve curriculum evolution as an operational reality within these schools. However, it is closely, perhaps inextricably, involved with various other programs, which will be described in this section.

A. Developing the "Creative Learning Experiences." As discussed earlier, an essential feature of the Madison Project's work is devising fully-worked-out classroom lessons (the so-called "creative learning experiences"). Some of these were developed as early as seven or eight years ago; others are being developed at the present time. Some have been developed without sponsorship; others (primarily those relating mathematics to science, or to individualized study involving physical apparatus) are developed as one part of a program sponsored by the Course Content Improvement section of the National Science Foundation; still others (primarily at the level of nursery school, kindergarten, and grades one and two, and for grades 9 and 10) are being developed as part of the present USOE demonstration project.[1]

B. Videotaping and Making Films. A major Project effort consists of recording actual classroom lessons on videotape and on 16 mm. film.[2] The process of recording these lessons is to interfere as little as possible with the ordinary conduct of the classroom, although some effort is made to try to record those classroom lessons which will be of most interest to teachers who will view the films. The films are intended for teacher education, and enable teachers to observe how actual classroom lessons might proceed.

Films (either made from videotape by the "kinescope" or "transfer-to film" process, or else made directly with 16 mm. film cameras) go through several stages: first, they are viewed and discussed by Project personnel to determine their probable value, strengths, and weaknesses; second, they are used experimentally with a few schools and colleges; third, if they appear to merit it, they are released to the general public.[3]

For the most part, films are not "edited" at all, since their authenticity as actual recordings of actual classroom lessons must not be compromised. There are one or two exceptions. Any film may be cut to determine the beginning and end of the lesson, without making any changes whatsoever in the portion between the cuts (that is to say, in the portion seen by the viewer). This is no more drastic then visting a class in a school -- observation must start somewhere, and terminate somewhere, but what is viewed in between is entirely authentic. The purpose of making beginning and end cuts is to save the viewers' time (since actual lessons may run from 30 minutes to as much as an hour or more in length), and to protect the viewer from being bombarded, in a single exposure, by more topics than he can easily comprehend.

---

[1] Davis (A 1964g) and Doris Diamont Machtinger, Experimental Course Report/Kindergarten. (Report No. 2, July, 1965), the Madison Project (in preparation).

[2] More recently, some films are being prepared as 8 mm. cartridge for use with the Fairchild Mark IV cartridge projector.

[3] Cf. Newsletter No. 1 (July, 1965), available from the Madison Project.

There are other exceptions. Films may, in rare instances, be cut by deleting portions of a lesson where the missing portion can be easily inferred from the part that is seen (a device common in dramatic presentations where one sees, say, a person enter a taxicab in front of building A, and alight from it in front of building B, etc.). Whenever this is done it is clearly indicated by titles included in the film: for example, by showing on the screen the sentence, "The following are excerpts from a 10-minute class discussion," and in addition by using forms of dramatic dissolves that make the transitions entirely unmistakable. (Cf., for example, the film entitled Graphing a Parabola.)

Further, some films use only short excerpts of lessons, perhaps accompanied by narrative. (Cf. the films Education Report: The New Math and Creative Learning Experiences.)

Finally, one film (The Tower Puzzle, a lesson taught by Donald Cohen and Ogie Wilkerson) was recorded directly on 16 mm. film, without the use of videotape. This film, like all films made in this way, had of course to be "cut together" in the usual cutting room procedure.

C. The Webster College Undergraduate Program. Webster College has since September, 1961, operated a college undergraduate program for the pre-service education of elementary school mathematics specialists, or K - 12 mathematics specialists. This program, financed in part by the Ford Foundation, was originally unique in the United States. Among its features:

i) Students who enter the program must usually have completed a full grade 9 through grade 12 mathematics program in high school (although exceptions may be made);

ii) Students may begin working in school classrooms as teacher assistants as early as their freshman year;

iii) Students may work on Webster College curriculum projects (the Madison Project, or WIMSA, which will be described below), and are encouraged to find summer employment with curriculum projects across the nation; the Karplus project at Berkeley, California; MINNEMAST., projects at the University of Illinois; projects at E.S.I.; and so forth.

iv) As the preceding may indicate, Webster students are treated more as apprentices or junior colleagues in instruction and curriculum revision, rather than as "college students";

v) Capable Webster students even conduct workshops for experienced teachers; although this may seem an unlikely venture, selected Webster students have done this with admirable success (and were not regarded as "college students" by the teacher-participants in the workshop):

vi) The Madison Project methods and materials are used prominently in the Webster program;

vii) "Official" student teaching is done at a variety of grade levels, from nursery school to grade twelve, and is spread over a period of several years in the student's career;

viii) Capable Webster students may extend their four-year bachelor's degree program, and include in it a regular teaching assignment in the Webster College Experimental School (which is not used for ordinary practice teaching);

ix) The program of mathematical studies for these students includes 30 credits of college-level mathematics; students are also encouraged to study physics, chemistry, biology, and the behavioral sciences;

x) The programs in mathematics, physics, biology, and behavioral sciences at Webster are coordinated to an unusual degree, and focus a considerable part of their emphasis on a coordinated approach to curriculum evolution and modern processes of instruction. Underlying part of this is emphasis on "inductive modes of learning," an orientation influenced heavily by Professor Jerome Bruner of Harvard University.

The preceding brief enumeration should make it clear that the Webster College program is unusual; actually, it is more unusual than has been indicated, and has been widely admired by visitors from all parts of the United States, and from abroad.

Under the administrative leadership of Sr. M. Francetta Barberis, S. L., and Sr. M. Jacqueline Grennan, S.L., and (as has been mentioned) with considerable guidance from Professor Jerome Bruner of Harvard, a remarkable program is being evolved. Two features, mentioned above, deserve further elaboration:

The coordination of the mathematics, physics, biology, and behavioral sicence programs, with a consequent focus on modern curriculum revision efforts and modern instructional procedures, is of central importance. A Webster student may, for example, teach a fifth grade class the concepts of slope and intercept using Madison Project materials, may perform velocity and acceleration experiments on automobiles and analyze the data using graphical differentiation and integration, may study the mathematical theory of differentiation and integration, may teach a unit on spiders to another fifth grade class, and may study the teaching of these classes from the point of view of behavioral scientists. She may even teach the theory of limits of infinite sequences to a high school class, and perform P.S.S.C. velocity and acceleration experiments with a junior high school class. At Webster College, things fit together.

Moreover, "things" at Webster are based upon the college student doing something worthwhile. The Bruner "inductive learning" approach, as used at Webster, amounts roughly to doing something (whether performing velocity and acceleration studies on automobiles, or teaching a unit on spiders to fifth graders), and then discussing what was done. Extensive abstract discussions not related to actual student experience are kept to a minimum. Discussions of teaching a unit on spiders to fifth graders -- which will have involved considerable work with actual living spiders -- will include both discussions of the relevant biology of spiders, and also discussions of behavioral science aspects of the teaching and learning of the students. But the root of all the discussions was an active involvement with reality -- the spiders were real, the children were real, and the Webster College student taught the unit (and probably even devised it) herself.

D. The Webster College MAT program has been discussed earlier in this report. It plays an essential role as the most advanced form of study of Madison Project methods and materials that is available anywhere, for elementary school teachers with no previous mathematics background.

E. The Syracuse University NSF Summer Institute Program. This program, directed by Professor Robert M. Exner of the Syracuse University Mathematics Department, gives secondary school teachers their best opportunity for learning the methods and materials of the Madison Project.

F. WIMSA -- The Webster Institute for Mathematics, Science, and the Arts -- is a center for curriculum and instruction work located at Webster College, which provides curriculum revision research opportunities for faculty and students that parallel those provided by the Madison Project, but includes all areas of the curriculum.

G. The "Specialist" Student-Teaching Program at Overland Park, Kansas. Under the leadership of Mrs. Adelyn Muller, a new student-teaching program for prospective elementary school teachers is available to students at several colleges and universities in the Kansas City area. The program provides both student-teaching experience as a self-contained classroom teacher, and also as an elementary school mathematics "visiting specialist" teacher. The "visiting specialist" portion of the program uses Madison Project materials.

H. The Experimental Project on the Use of Cartridge Projectors. Through Mrs. Beryl S. Cochran, the Madison Project has been cooperating in an experimental project at Teachers College, Columbia University, on the use of cartridge projectors.

I. Training Classroom Observers. The Madison Project has cooperated with several experimental projects that use films to train classroom observers. Much systematic observation of classroom lessons in the past has restricted observers to making only the most rudimentary sorts of discrimination -- for example, to recording whether the teacher was talking, or a student was talking, or neither.

Much deeper insight into the real nature of classroom lessons can be gained if observers can be trained to make for more subtle types of discriminations. The use of films showing actual classroom lessons may provide one way to train observers in these skills. It is important here that the films are not "scripted" or "acted." They show actual lessons as these lessons would ordinarily occur. Consequently judgements on enthusiasm, interest, degree of involvement, anxiety level, etc., can be attempted.

J. Finally, the Madison Project has recently become privileged to cooperate with the Nova School, in Fort Lauderdale, Florida. By re-allocating money, time, and effort Nova has been able to create the finest mathematics program, at a pre-college level, anywhere in the United States, in a school with an annual per-student cost of $400. Director of the mathematics program is Mr. Burt Kaufman. The Nova re-allocation focuses on curriculum and instruction. The Nova excellence in these areas is achieved in part by what Nova does not do:

i. They do not operate a catering service. They have built no cafeteria. Students arrange for their own lunches.

ii. They do not operate a transportation service. Students get between home and school by their own devices (or their parents').

iii. They do not operate a baby-sitting service, nor provide custodial care. If a student's class ends at 9:00 A.M., and his next class is at 10:30 A.M., then he is on his own from 9 until 10:30. Laboratories and other facilities are also used by students, unattended by adults. The Nova philosophy is that if you can't trust your students, you won't be able to contribute much to their education anyhow.

iv. They do not teach the obvious, or the trivial. Bright students do not take courses to learn how to factor polynomials or to study trigonometry. They do this on their own. In their courses they study Birkhoff and McLane (or its approximate modern equivalent), Moise's Elementary Mathematics from an Advanced Viewpoint, and a modern approach to mathematical analysis of real variables.

v. They do not take long vacations. Students attend Nova eleven months a year.

VIII. MATERIALS PRODUCED

Materials produced, either by the Madison Project, or by individuals identified with the Project, include the following:

1. J. Robert Cleary, "A Study of Test Performance in Two Madison Project Schools and One Control School," 1965, available from the Madison Project.
2. Gordon H. Clem, "The Madison Project," report of the Annual Conference of the National Association of Independent Schools (June, 1963), pp. 46 - 49.
3. Beryl S. Cochran, "The New Math," Connecticut College Alumnae News (December, 1962), pp. 11 - 13.
4. Donald Cohen, "A Lesson on Absolute Value," The Arithmetic Teacher, Vol. 11, No. 8 (December, 1964), pp. 561 - 562.
5. Robert B. Davis, "Emotion and Thought," The Mathematics Teacher, Vol. XLVIII, No. 3 (March, 1955), pp. 133 - 142 (reprinted in M. Vere Devault's Improving Mathematics Programs, Merrill Books, Columbus, Ohio, 1961, pp. 108 - 121).
6. Robert B. Davis, "The Mathematics Revolution: Causes and Directions," Frontiers of Secondary Education, Syracuse University Press (1958), Chapter 7.
7. Robert B. Davis, "How Do We Learn Mathematics?" MIT Technology Review, Vol. LXII (1959), pp. 28 - 30.
8. Robert B. Davis, "Seventh Grade Algebra: The Madison Project," New York State Mathematics Teachers Journal, Vol. IX, No. 2 (1959), pp. 5 - 8.
9. Robert B. Davis, Solving Problems and Constructing Systems -- Quadratic Equations and Vectors, report of an Orientation Conference for SMSG Experimental Centers, Chicago, Illinois, September 19, 1959, pp. 97 - 101.
10. Robert B. Davis, "Special Report -- The Road Between," Updating Mathematics (October, 1959), Section 1.
11. Robert B. Davis, "The Syracuse University 'Madison Project'," American Mathematical Monthly, Vol. LXVII, No. 2 (February, 1960), pp. 178 - 180.
12. Robert B. Davis, "Mathematics For Younger Children -- The Present Status of the Madison Project," New York State Mathematics Teachers Journal, Vol. X, No. 2 (April, 1960), pp. 75 - 79.

13. Robert B. Davis, "The 'New' Pre-College Mathematics," Undergraduate Physics and Mathematics in Electrical Engineering, Proceedings of the Sagamore Conference, September, 1960, Department of Electrical Engineering, Syracuse University.

14. Robert B. Davis, "Algebra in Grades Three Through Ten: The Madison Project," Grade Teacher, Vol. LXXVIII (October, 1960), pp. 34 +.

15. Robert B. Davis, "The Madison Project: Algebra in Grades 3 - 9," Frontiers of Elementary Education, Syracuse University Press (1961), pp. 56 - 59.

16. Robert B. Davis, Axioms for Arithmetic and Algebra, the Madison Project, 1962.

17. Robert B. Davis, A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project (1962). Report submitted to the Commissioner of Education, U. S. Department of Health, Education and Welfare.

18. Robert B. Davis, The Madison Project: A Brief Introduction to Materials and Activities, the Madison Project (1962 ; revised 1965).

19. Robert B. Davis, Notes on the Film: A Lesson With Second Graders (booklet to accompany film of the same name), the Madison Project, 1962.

20. Robert B. Davis, Notes on the Film: First Lesson (booklet to accompany film of the same name), the Madison Project, 1962.

21. Robert B. Davis, Notes on the Film: Matrices (booklet to accompany film of the same name), the Madison Project, 1962.

22. Robert B. Davis, "Algebra in Grades Four, Five and Six," Grade Teacher, Vol. LXXIX (April, 1962), pp. 57 +.

23. Robert B. Davis, "Report on the Madison Project," Science Education News (December, 1962), pp. 15 - 16 (a publication of the American Association for the Advancement of Science).

24. Robert B. Davis, A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project, 1963, report submitted to the Commissioner of Education, U.S. Department of Health, Education and Welfare.

25. Robert B. Davis, "The Evolution of School Mathematics," Journal of Research in Science Teaching. Vol. 1 (1963), pp. 260 - 264.

26. Robert B. Davis, "The Syracuse University - Webster College Madison Project," in An Analysis of New Mathematics Programs, edited by Philip Peak, et al. NCTM (1963), pp. 16 - 19; 20 - 21.

27. Robert B. Davis, "Mathematical Thought and The Nature of Learning: The Madison Project View," Frontiers of Education. Report of the Twenty-Seventh Educational Conference sponsored by the Educational Records Bureau. American Council on Education, Educational Records Bureau, 21 Audubon Ave., New York 32, New York (1963), pp. 79 - 83.

28. Robert B. Davis, Matrices, Functions, and Other Topics: A Student Discussion Guide, the Madison Project, 1963.

29. Robert B. Davis, Matrices, Functions, and Other Topics: A Text for Teachers, the Madison Project, 1963.

30. Robert B. Davis, "Three Periods of Education in the United States," Christian Science Monitor (January 12, 1963), p. 11.

31. Robert B. Davis, "Math Takes a New Path," The PTA Magazine, Vol. LVII, No. 6 (February, 1963), pp. 8 - 11.

32. Robert B. Davis, "How Is The Madison Project Different?" Illinois Council of Teachers of Mathematics News Letter, Vol. XIV, No. 2 (May, 1963), pp. 7 - 8.

33. Robert B. Davis, "Report on Madison Project Activities, September, 1962 - November, 1963." A report submitted to the National Science Foundation (December, 1963).

34. Robert B. Davis, Discovery in Mathematics: Student Discussion Guide, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964.

35. Robert B. Davis, Discovery in Mathematics: A Text for Teachers, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964.

36. Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, Vol. 2 (1964), pp. 146 - 162.

37. Robert B. Davis, "Undergraduate College Education in Relation to the 'New Curriculum' Projects," Current Issues in Higher Education (1964), pp. 153 - 155.

38. Robert B. Davis, "Report on the Syracuse University - Webster College Madison Project," American Mathematical Monthly, Vol. 71, No. 3 (March, 1964) pp. 306 - 308.

39. Robert B. Davis, "Some Remarks on a Theorv of Instruction," Piaget Re-discovered. Edited by Verne N. Rockcastle and Richard E. Ripple. A report of the Conference on Cognitive Studies and Curriculum Development (March, 1964). School of Education, Cornell University, pp. 134 - 138.

40. Robert B. Davis, Experimental Course Report/Grade Nine (Report No. 1, June, 1964), available from the Madison Project.

41. Robert B. Davis, "Guidelines for the Assessment, Initiation, and Reassessment of Curriculum Programs" (paper presented at the Chicago ASCD meeting, Fall, 1964).

42. Robert B. Davis, "Goals for School Mathematics: The Madison Project View," Journal of Research in Science Teaching, Vol. 2, Issue 4 (December, 1964), pp. 309 - 315.

43. Robert B. Davis, "Mathematics," Chapter 6 in: Glenys Unruh, editor, New Curriculum Developments, ASCD (1965), pp. 46 - 56.

44. Robert B. Davis, "Some Remarks on 'Learning By Discovery'," available from the Madison Project (1965).

45. Robert B. Davis, "Recent Activities of the Madison Project," American Mathematical Monthly, Vol. 72, No. 1 (January, 1965), pp. 72 - 73.

46. Robert B. Davis, "New Mathematics Is Deeper Than Mathematics," Five Views of the "New Math," Council for Basic Education, Washington, D.C. (April, 1965), pp. 10 - 12.

47. Robert B. Davis, Matrices, Logic, and other Topics: Students' Discussion Guide (in preparation), Addison-Wesley Publishing Company.

48. Robert B. Davis, Matrices, Logic, and other Topics: A Text for Teachers. (in preparation), Addison-Wesley Publishing Company.

49. Sheldon Gold, The Reaction of Teachers to the Methods of Teaching Modern Mathematics Using Madison Project Materials. School of Education, the City College of the City University of New York, January, 1965.

50. Charles D. Hopkins, "An Experiment on Use of Arithmetic Time in the Fifth Grade" (unpublished doctoral thesis), School of Education, Indiana University (June, 1965).

51. In-Service Course I, the Madison Project, 1964.

52. In-Service Course II, the Madison Project, 1965.

53. Katherine Kharas, et al., Mathematics. Prepared at the Entebbe Mathematics Workshop. Secondary Two, Vol. 3, March, 1964. African Education Program, Educational Services Inc., Watertown, Massachusetts.

54. Doris Diamont Machtinger, Experimental Course Report/Kindergarten. (Report No. 2, July, 1965), the Madison Project (in preparation).

55. Doris Diamont Machtinger, Experimental Course Report/First Grade, the Madison Project (in preparation).

56. Newsletter No. 1 (July, 1965), the Madison Project.

Individualized Study Materials:

57. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 1, Peg Game, the Madison Project (in preparation).

58. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 2, Weights and Springs, the Madison Project (in preparation).

59. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 3, Tower Puzzle, the Madison Project (in preparation).

60. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 4, Pulley, the Madison Project (in preparation).

61. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 5, Pendulum - 1, the Madison Project (in preparation).

62. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 6, Conic Sections - 1, the Madison Project (in preparation).

63. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 7, Geoboard, the Madison Project (in preparation).

64. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 8, Conic Sections - 2, the Madison Project (in preparation).

65. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 9, Pendulum - 2, the Madison Project (in preparation).

66. Donald Cohen, Independent Exploration Materials/Grades 5 - 9, No. 10, Geometry - 1, the Madison Project (in preparation).

Films released for general distribution:

First Lesson	(sponsor: Notre Dame University)
Second Lesson	(sponsor: Notre Dame University)
A Lesson with Second Graders	(sponsor: Webster College)
Guessing Functions	(sponsor: USOE, under contract No. D-093)
Graphing a Parabola	(sponsor: NSF)
Audiotape D-1	(sponsor: Marcel Holzer Foundation and Alfred P. Sloan Foundation)

Videotapes and films recorded, but (except for those listed above) not yet released for general distribution:

i. Films involving Class "A"[1]

The first class we shall discuss we shall designate Class "A." Nametags on students read "Lex," "Bruce," "Ann," "Sarah," "Debby H.," "Ellen," "Geoff," "Jeff," etc. In December, 1963[2] this class was in the 9th grade, and were studying S.M.S.G. "10th grade" geometry (independently of the Madison Project). They also met with 8th and 7th graders (from classes "B" and "C" as discussed below) in voluntary "Saturday Seminars" devoted to straight-line kinematics experiments, conservation of momentum experiments, the programming of digital computers, and other topics (under Madison Project auspices).

1. The audiotape "$\square + \square = 2 \times \square$" (#D-1). The first appearance of Class "A" in Project materials occurs not in a film, but in the audio-tape entitled "$\square + \square = 2 \times \square$." Class "A" were 5th graders when this lesson was recorded (all classroom mathematics lessons of Class "A" were recorded -- on audio-tape --during their years in grade 5 and grade 6;

---

[1] Because the Project attempts in many cases to follow the same students from one year to the next, over as long a span of years as possible, it is worth identifying the classes of students involved. This is done by designating the various classes using letters of the alphabet, in (roughly) chronological order.

[2] At the time that most of the notes were made, for inclusion in the December 16, 1963 report to the National Science Foundation. In the Present report, additional notes have been added to include films made prior to September, 1965.

the transition from audio-tape to films, involving vastly greater costs, required the Project to become selective, and to film only occasional classes, usually about 4 times per year). Prior to the classroom lesson on tape No. D-1, the students had developed an extended list of algebraic identities, including for example

$$\square + \triangle = \triangle + \square$$

$$\square + \square = 2 \times \square$$

$$(\square + \square) + \square = 3 \times \square$$

$$\square \times 1 = \square$$

$$\square + 0 = \square$$

$$(\square + \triangle) - \triangle = \square$$

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + \left[(\triangle + \triangle) \times \square\right] + (\triangle \times \triangle)$$

and many others. They had been asked to devise methods for shortening such lists, and had devised two methods: first, generalization, by using variables, so that

$$(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$$

$$(\square + 4) \times (\square + 4) = (\square \times \square) + (8 \times \square) + 16$$

$$\vdots$$

was condensed (by Lex) into the single identity

$$(\square + \triangle) \times (\square + \triangle) = (\square \times \square) + \left[(\triangle + \triangle) \times \square\right] + (\triangle \times \triangle).$$

Secondly, they had worked out the method of shortening lists by means of inference or implication, via algebraic derivations.

The class was now asked to determine the shortest list from which all their other identities could be derived. At this point, the lesson recorded on tape No. D-1 took place, in the late fall of 1959. Discussion centered upon the issue of whether the identity $\square + \square = 2 \times \square$ should be classified as an axiom (i.e., retained on the minimal list) or as a theorem (i.e., whether it could be obtained by a derivation from statements already on the "axiom" list). The students settled the matter by making up two original derivations for the identity $\square + \square = 2 \times \square$.

2. (no official title) "Complex Numbers." Class "A," as 6th graders, are recorded

in a lesson which represents their first encounter with isomorphism, with the isomorphism between rational numbers and a subset of the set of 2-by-2 matrices, and with the equation

$$x^2 = {}^{-}4$$

While the mathematical and educational content of this film is exciting, the technical aspects of the television recording are poor. It was consequently necessary to return to this topic again the following year, in order to obtain a technically more satisfactory film ("Complex Numbers via Matrices"), which was at the same time inevitably more sophisticated mathematically, since the mathematical growth of the students during the intervening time could not be denied.

3. (no official title) "Inductive Sets." Class "A," as 6th graders, in a Madison Project adaptation of David Page's idea of "hidden numbers." The class has the task of determining a set of numbers from a set of clues. The clues are not necessarily all correct, and may contain contradictions. By identifying the contradictions, the class is able (under established "rules of the game") to compel the teacher to identify the clues as "True" or "False." One clue states: "The set A is inductive (i.e., $x \in A$ implies $x + 1 \in A$)." Here, again, the mathematical and educational content is good, but the film quality is relatively poor. This film, shot during the academic year 1960-61, is one of the first films made by the Project, with funds provided by Webster College.

4. (no official title) "Derivations." Class "A," as 6th graders, prove the algebraic identity

$$\square + (\square + \square) = 3 \times \square$$

from a suitable set of axioms, consisting primarily of these: closure of + and x;

$\forall_{x,y}$, $x + y = y + x$; $\forall_{x,y}$, $xy = yx$; $\forall_{x,y,z}$, $x(y + z) = xy + xz$;

$\exists$ element 1 such that $\forall_x$, $x \cdot 1 = x$; $\exists$ element 0 such that, $\forall_x$, $x + 0 = x$;

numerals are defined recursively, by $1 + 1 = 2$, $2 + 1 = 3$, $3 + 1 = 4$, ...;

$\forall_{x,y,z}$, $(x + y) + z = x + (y + z)$; $\forall_{x,y,z}$, $(xy)z = x(yz)$;

plus the usual uses of replacement with variables, and substituting of one name for another name for the same thing. The class involves lively, excited class discussion; the lesson is not perceived by the children as excessively formal, by any means, although the preceding description may make it sound that way. Perhaps the answer is that the mathematics has a certain precision, and is in that sense formal, but the classroom atmosphere is informal, lively and exciting.

5. (no official title) "Determining the Height of a Ladder." By using ratio and pro-

portion, scale drawing, similar triangles, and angle measurement by sighting along a blackboard protector, the students, working mostly in groups of 3 or 5, determine the height of a ladder as a problem in indirect measurement. After all the computed heights are reported, some students climb the ladder and measure its height directly. (Class "A," as 6th graders.)

6. (no official title) "Derivations and Oppositing." Class "A," as 7th graders, proving various theorems in axiomatic algebra (for rational numbers), including some involving additive inverses.

7. (no official title) "Yarn-Breaking Experiments." Class "A," as 7th graders, using vector methods to predict the breaking or non-breaking of wool and nylon yarns under various conditions of loading with 10 lb. and 20 lb. lead weights. After making their calculations, the students perform the experiments to test their predictions.

8. (no official title) "Forces and Vectors." This film consists of some excerpts from the film "Yarn-Breaking Experiments." (No. 7, immediately above).

9. Education Report: The New Math. This film was made by WNBC, Channel 4, New York City. It includes many short sections of films involving Class "A," including some excerpts from film No. 7 listed above.

10. Graphing an Ellipse. The film Graphing an Ellipse records a classroom lesson that occurred in the late fall of 1961, when the students in Class "A," were 7th graders. Prior to this lesson, they were familiar with graphs of linear functions, with the concepts of slope and intercept, with graphs of circles, and with graphs of parabolas. In this lesson they meet ellipses for the first time, in the case of

$$x^2 + 4y^2 = 25,$$

and, for the first time, they are asked to use graphs to study a relation involving a parameter k, in the example

$$x^2 + ky^2 = 25, \qquad 0 \leq k.$$

11. Complex Numbers via Matrices. Filmed at about the same time as Graphing an Ellipse, this film is concerned with comparing different number systems, and identifying equations for which, within a given number system, the truth set is the empty set (e.g., $3 + \square = 3$ within the system of positive integers). The number systems are extended to provide non-empty truth sets for these equations where possible. The class explores the question of whether a system of elements $\alpha, \beta, \gamma, \delta, \ldots$ is isomorphic to a specified familiar number system or not. An isomorphism between rational numbers and matrices of

the form

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

is established, and used to translate

$$x^2 = {}^{-}4$$

into matrix language. Solutions for this matrix equation are then explored, and the closure of the relevant subsets of the set of 2-by-2 matrices is investigated. This lesson is marked by vigorous class discussions at many points. (Class "A" had had some previous experience with the equation

$$x^2 = {}^{-}4$$

during the preceding year, when they were 6th graders. Cf. film No. 2, above).

12. Report to the Miami Meeting. A shorter version of film No. 11 above. The shortening is accomplished by deleting the first part of the lesson, dealing with isomorphisms and extending number systems. The later part of the lesson, dealing with matrices and complex numbers, is retained exactly as the lesson occurred in class.

13. (no official title) "Derivations of the Quadratic Formula." Class "A," as 7th graders, making an original derivation of the quadratic formula. (Filmed in the fall of 1961.)

14. (no official title) "Slope of a Curve and Graphical Differentiation." Class "A," as 8th graders.

15. (no official title) "Limit of a Sequence, and Irrational Numbers." Class "A," as 8th graders, studying limit of a sequence, and the problem of extending the rational number system so as to find roots for

$$x^2 = 2.$$

The key is a new axiom proposed by a student (Jeff), stating that every bounded monotonic sequence has a limit.

16. (no official title) Limit of a Sequence, with Dr. Ross Finney of M.I.T. The purpose of this experimental film was to record the students' reactions to a mathematician

not familiar to them, and not involved with the Madison Project. The subject was limit of a sequence. Filmed in the academic year 1962-63, when Class "A" was in 8th grade.

17. (no official title) Proof of the Theorem that

$$\lim_{n \to \infty} (u_n + v_n) = \lim_{n \to \infty} u_n + \lim_{n \to \infty} v_n \ .$$

Class A, as 8th graders. (Spring, 1963).

18. (no official title) "Velocity and Acceleration." This film is perhaps a deviation from the general practice of filming "actual classroom lessons," since it was made as follows: A class of 8th graders "Class A," who had at the time been studying Madison Project materials since grade 5), who were quite familiar with graphs of functions, and slope of linear graphs, and who had some familiarity with the concepts of limit of a sequence, tangency, rate of change, and derivative, were allowed to view the PSSC film Straight Line Kinematics as often as they wished, either during mathematics class or after school. Each student was allowed to select whatever equipment he wished from among the standard PSSC kinematics apparatus, and to take this equipment home if he chose. Either in school, or outside of school, each student could devise, perform, and analyze such experiments as he chose. After several weeks of such activity, mostly outside of school, the students came in, set up their apparatus, performed experiments, and analyzed their data, in full view of television cameras. The resulting film has not as yet been edited, nor given titles, but two excerpts from it are included in the film entitled Creative Learning Experiences, film No. 18, described immediately below.

19. Creative Learning Experiences. This film was made originally for presentation at a conference on juvenile delinquency, dropouts, and school failure. It consists of a sequence of short excerpts from various other films (including No. 18 immediately above), and was intended to suggest that greater diversity in school activities might cause school to have more appeal for more children. "Traditional" school programs are very abstract, "bookish," language-oriented affairs -- "school" is a matter of talking and listening and reading and writing. This has little appeal to some children. But school need not be like this.

This film was also presented at the U. S. Office of Education Demonstration Center in Washington, D. C., on November 15, 1963.

20. (untitled film, made in West Hartford, Connecticut, January 11, 1964) A lesson in modern 10th grade geometry taught to Class "A," who were at that time 9th graders. The teacher is Mrs. Elizabeth Herbert. The purpose of the film was to help answer the question, "What happens to children who have studied Madison Project materials for several years, and who then find themselves in some other program?" In fact, Class "A" began the study

of Madison Project materials in grade 5 (in fall, 1959), and continued for four years, through the end of grade 8. At that point their study of Madison Project materials ceased, and they shifted to S.M.S.G. materials. In grade 9 they studied 10th grade S.M.S.G. geometry, with Mrs. Herbert as teacher. This film records the result, in the case of one long lesson lasting $1\frac{1}{2}$ hours.

21. (untitled) Shorter version of film No. 20.

ii. Films Involving Class "B"

Nametags on students in Class "B" read "Beth," "Jean-Anne," "Toby," "Mark," "Flint," etc. In December, 1963, this class was in the 8th grade. Class "B" began studying Madison Project materials when they were in the 4th grade, in the academic year 1959-1960. During their first year of study, no films were made, but all classes were recorded on audio-tape.

Class "B" appears in films alone, and also in films where some students from Class "B" are combined with some students from Class "C," who are one year younger. This combination was made for several reasons, but especially to emphasize that placement of "modern mathematics" topics by grade level is not unique nor fully determined, and to emphasize the educational desirability of "crossing grade lines," as in Stoddard's "dual progress plan," etc. The inappropriateness of dividing children into "grades" shows up especially strongly when one attempts to build a high-quality "modern" mathematics program. "Grades" were presumably introduced primarily to solve administrative problems of coping with vast numbers of children in our schools, but they are distinctly a device for mass production, and are not suitable for high quality education.

22. (Audiotapes.) While Class "B" were 4th graders, their mathematics classes were recorded on audio-tapes, using professional multi-channel recording equipment (academic year 1959-1960). Their mathematical work in 4th grade followed the pattern of the book Discovery in Mathematics.[1]

23. (no official title) "Hidden Numbers." This, again, is a Madison Project adaptation of David Page's important notion of "Hidden Numbers." This film was made during the academic year 1960-1961, when the students were in the 5th grade. The mathematical content is similar to that described in film number 3 listed above.

---

[1] Davis (A1964 ab).

24. (no official title) "Truth Sets for Quadratic Equations." This film was made in 1960-1961, Class "B" then being in grade 5. The children use the coefficient rules for polynomial equations (which, in the case of quadratic equations in normal form they have discovered for themselves) to find truth sets for quadratic equations.

25. (no official title) "General Form of an Equation, and General Solution of an Equation." Made in 1960-1961, Class "B" then being 5th graders. Work with equations of the form

$$\square + 3 = 7$$

is generalized to

$$\square + a = b,$$

with truth set

$$\{b - a\}, \text{ etc.}$$

Similarly for

$$(a \times \square) + b = c, \quad (a \times \square) + (b \times \square) = c, \quad (a \times \square) + \square = c,$$

and so on.

26. Matrices. This film shows some students from Class "B," combined with some students from Class "C," who are one year younger. The film was made in the 1961-1962 academic year, when the students in Class "B" were 6th graders, and those in Class "C" were 5th graders.

Prior to the film, the students in both classes had, separately, studied an axiomatic approach to the algebra of rational numbers, following the general pattern described earlier for Class "A." The experiences of the two classes prior to filming were not completely identical; this can be seen in the film when, for example, one class is familiar with the commutative laws, and the other is not.

Also prior to the film and after the axiomatic algebra just mentioned, both classes had been shown the "inner product" pattern for a covariant vector and a contravariant vector.

$$(A \ B \ C) \times \begin{pmatrix} D \\ E \\ F \end{pmatrix} = (A \times D) + (B \times E) + (C \times F).$$

They had then been given two tasks: extend this "inner product" pattern to larger arrays of numbers, which the teacher guided in such a way that 2-by-2 matrices ultimately were included; then explore the algebraic structure of this new algebraic system of 2-by-2 matrices (this latter was especially an exercise in trying to find suitable questions to ask. Some suitable questions, of course, would be: Is matrix addition commutative? Associative? Is matrix multiplication commutative? Associative? How about the distributive law? Is there an additive identity element? Are there additive inverses? Is there a multiplicative identity element? When are there multiplicative inverses? and so on.)

In the lesson shown on the film, the children review much of their past work with matrices. They then approach the question: "Is there a matrix that behaves like the number 0?" This question is new; without difficulty the students find that the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

"behaves like 0" for 2-by-2 matrices, in the sense that

$$\forall_A, \quad A + 0 = A$$

$$\forall_A, \quad A \times 0 = 0.$$

This leads naturally to the question "Is there a matrix that behaves like 1?" The obvious guess is that perhaps

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

will; trial, of course, shows that it does not satisfy a matrix equation of the type

$$\forall_A, \quad A \times 1 = A.$$

Here is a delicious dilemma: have we failed because there is no "1" matrix, or because we haven't yet been clever enough to find it?

Ricky, one of the 5th grade students, finds that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

behaves as required. In the course of this work, the children make various errors, as a result of which, for example, they find that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & A \\ D & C \end{pmatrix},$$

which becomes known as "Mary's reversing matrix." The children build on these errors, and extend their work in a very creative fashion leading them essentially toward the isomorphism between the rational number A and the 2-by-2 matrix

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

into questions of reversing rows and columns, into projection operators, into matrices such that

$$M^2 = I,$$

and so on. The film shows clearly how 5th and 6th graders can explore open-ended questions in mathematics.

27. (no official title) Programming the IBM 1620, using GOTRAN. As with film No. 26, immediately above, this film shows a class including students from Class "B" working together with students from Class "C." This film was made in 1961-1962, when Class "B" was in grade 6, and Class "C" was in grade 5. The children write original programs for the IBM 1620 digital computer, using GOTRAN. A 1620 computer installation in the school building itself was provided and staffed by I.B.M., at their expense.

28. (no official title) "Derivation of the Quadratic Formula." Class "B" in 6th grade (1961-1962), combined with Class "C," as 5th graders, deriving the quadratic formula. This derivation is spread over several lessons, involving a sequential attack via

$$x^2 = 16$$

$$x^2 = p \quad \text{(p is a perfect square)}$$

$$(x-2)^2 = 16$$

$$(x-2)^2 = p \quad \text{(p is a perfect square)}$$

$$(x-a)^2 = 144$$

$$(x-a)^2 = p \quad \text{(p is a perfect square)}$$

$$x^2 - 2ax + a^2 = p \quad \text{(p is a perfect square)}$$

$$x^2 - 2ax + b = p \quad \text{(with appropriate assumptions).}$$

The film listed as film No. 13 above ("Derivation of the Quadratic Formula") shows Class "A," as 7th graders, completing the final lesson of this sequence; the present film, with Classes "B" and "C" in grades 5 and 6, shows the first lesson in this sequence. As might be expected, film number 13 is distinguished by sophistication and power; film #28 is distinguished by naiveté and creative originality.

29. (no official title) "Introduction to Derivations." Class "B" in 6th grade, combined with Class "C" in 5th grade. This film was made in January, 1962. The familiar

type of word problem

TALL

PAIN,

with intermediate steps to be added, is used to introduce the idea of algebraic derivations in a "one-step-at-a-time" format. This format is then used to write a derivation of the theorem:

$$A + (B \times C) = (C \times B) + A.$$

There is a brief discussion of the role of "axioms" and "theorems." (Video Tape #VT-9, Part I)

30. (no official title) "Derivation of Geometric Mensuration Formulae." The same students as in the preceding lesson; taped at the same time. Assuming the formula for the area of a rectangle to be known, the students derive formulas for the areas of various other geometric figures, by "cutting and rearranging," etc., using the additive property of measure and some intuitive ideas of congruence. Formulas for perimeters are also developed. (Video Tape No. VT-9, Part I)

31. (no official title) "Word Problems." With the same students as film #28 above, and filmed at about the same time. The word problems follow the pattern of Discovery in Mathematics, and include simultaneous equations, inequalities, and divisibility criteria.

32. (no official title) "Rational Approximations to $\sqrt{2}$." The same students as in films #28 and #29, filmed at about the same time.

33. Introduction to Truth Tables and Inference Schemes. Class "B" again; this film was shot in the academic year 1962-1963, when the students were in 7th grade. It follows the general pattern of the work in logic described in the report of the Cambridge Conference on School Mathematics.[1]

34. (no official title) "Experience with Probability." Class "B," as 7th graders (1962-1963). The students are collecting empirical data on random events, following the general

---

[1] Goals for School Mathematics (B 1963), p. 38.

pattern of Frederick Mosteller's work (as in Continental Classroom, or Probability with Statistical Applications,[1] or the Report of the Cambridge Conference[2]).

iii. Films Involving Class "C"

Nametags on students in Class "C" read: "Jeff," "Ricky," "Mary," "Pam," "Lilli," "Windy," "Jane," "Geoff," "Greg," "Kris," "Val," "Miklos," "Jill," "Judy," "Jackie," "Tina," "Don," "Jennifer," etc. In December, 1963, this class was in the 7th grade. They began the study of Madison Project materials in the academic year 1959-1960, when they were in 3rd grade. During their 3rd grade and 4th grade years, no films involving Class"C" were successfully recorded, but all of their mathematics lessons were recorded on audio-tape, using multi-channel professional tape recording equipment. (Magnecorders, etc.) One film was attempted with Class "C" as 4th graders, but it was destroyed by a malfunction of the video tape recording equipment. Class "C" appears in films alone, and also jointly with Class "B," as discussed above.

35. Audiotapes. The mathematics lessons of Class "C" when they were 3rd graders (1959-1960) and 4th graders (1960-1961) are recorded on audiotapes.

36. Matrices. This is the identical film listed above as #26. Classes "B" and "C" appear together in this film. (Recorded 1961-1962, when Class "C" were 5th graders.)

37. Class "C" appears, together with Class "B," in the films listed above as #27, #28, #29, #30, #31, and #32. These films were made in the academic year 1961-1962, when the children of Class "C" were in 5th grade.

38. Clues. Made in November, 1962, when the students were 6th graders. This is an adaptation of David Page's notion of "Hidden Numbers." It is intended to provide experience with:

implication
contradiction

---

[1] Mosteller (D1961).

[2] Goals for School Mathematics (B 1963), p. 40.

existence
uniqueness
variables.

As in all films (and all lessons) with Class "C," this lesson is marked by vigorous, clever, creative class participation.

39. Accumulating a List of Identities. Class "C," in grade 6 (1962-1963). The Madison Project approach to algebra[1] begins with variables, open sentences, truth sets, and goes on to identities, which must at first be accepted tentatively by generalizing from many instances.

The students accumulate a long list of identities, and then attempt to use implication and generalization (using variables) to separate this long list in two sub-lists, one labelled "Axioms," and the other labelled "Theorems." This film shows one of the early lessons in this program: the very beginnings of the program can be seen in the films First Lesson, Second Lesson, and An Introduction to Identities, made by Class "D" and described below; later stages in this program are presented in audiotape #D-1, in the film Axioms and Theorems (#40 below), and in various unedited films made with Class "A" (#4 and #6 above).

40. Axioms and Theorems. Class "C," as 6th graders (1962-1963). This film continues the program described under #39 immediately above. (Cf. also #41 below.)

41. (no official title) Shortened version of "Axioms and Theorems." This film is a shortened version of #40 immediately above, prepared by Miss Widge Cochran.

42. Average and Variance. Class "C," 6th graders (1962-1963). Four children independently guess the length of the classroom; for these 4 numbers the average and the variance are computed. Four children independently measure the length of the classroom, using 6-inch plastic rulers; for these 4 numbers the average and variance are computed. Four children independently measure the room, using meter sticks; an average and variance are computed. Finally, four teams of children independently measure the length, using high-quality surveyor's tape measures, and the average and variance for these four numbers are computed. Throughout there is discussion of what is going on, and why.

43. Weights and Springs. Class "C," 6th grade, 1962-1963. Study via graphs and algebraic equations, of functions obtained empirically. (In this case, a linear function.) In the filmed lesson, the teacher has a single apparatus set up in front of the class. Many viewers have suggested that a better procedure would probably be to break the class up into

---

[1] CF. Davis (A1964ab), and the audio-tape #D-1 entitled "□+□= 2×□."

"committees" of 3 (or so) children each, and give each committee its own apparatus. The teacher would then visit with each of the separate committees.

44. Solving Equations with Matrices. Class "C," grade 6, academic year 1962-1963. The content is generally similar to the film described above as #2. However, whereas the technical film quality (as distinct from the quality of the lesson itself) of film #2 is unsatisfactory, the technical film quality of film number 44 is quite satisfactory. The language used in the lesson on film #44 is slightly less sophisticated than that used in the lesson on film #2 (for example, the concept of "isomorphism" is not explicitly introduced in the lesson on film #44).

45. (no official title) Paul Merrick's Film on Gas Laws. Class "C," grade 6, academic year 1962-1963. The teacher in this film is Paul Merrick, of Educational Services Incorporated, Watertown, Massachusetts.[1] The subject is the unit on gas laws developed by E.S.I.'s Elementary School Science Project, under the direction of Professor David Hawkins. This film was made in cooperation with E.S.I.

46. Gases and Airs in the Classroom. This film is an edited and modified version of film #45 immediately above. It was produced by E.S.I., and directed by Jack Churchill. It is not available through the Madison Project; for information, write Educational Services Incorporated, 47 Galen Street, Watertown 72, Massachusetts.

47. (untitled film from videotape #VT-56) A lesson in logic, taught by Robert B. Davis, on March 13, 1965, when Class "C" were in grade 8. The version of logic used in this lesson is a rather more polished version than that used earlier, and reflects the experience gained in teaching logic to elementary school, high school, and college students over the past seven years.[2]

48. (untitled film from videotape #VT-57) A continuation of the lesson on logic which begins on VT-56. Cf. item #47 immediately above.

iv. Films involving Class "D"

Class "D" is unusual in several respects. All filming involving Class "D" occurred in a single weekend, during the summer of 1960. Thus, Class "D" is not one which has been followed for several consecutive years, according to the normal practice of the Madison

---

[1] Professor Merrick is now (September, 1965) at Webster College.

[2] For technical details, please consult the Quarterly Reports dated March 31, 1965 (p.2) and June 30, 1965 (p.9).

Project. Also, Class "D" was filmed in a television studio. (WNDU-TV, South Bend, Indiana), rather than in a regular elementary school, as is usually done in Project films. Finally, Class "D" consists of children from grades 3 through 7; thus it is an extreme demonstration of the potential for "crossing grade lines" or for an "ungraded" mathematics program (as in Stoddard's "dual progress plan, " etc.) The producer for the films involving Class "D" was Professor Arnold Ross, then Chairman of the Mathematics Department at Notre Dame University; the teacher was Professor Robert B. Davis, of Syracuse University; the content of these lessons is standard Madison Project repertoire.[1]

49. First Lesson. This shows an actual first lesson of Madison Project materials, and introduces variables, open sentences, truth sets, signed numbers, and Cartesian co-ordinates. The lesson is described in detail in the booklet entitled Notes on the Film: First Lesson.

50. Second Lesson. A continuation, on the next day, of the work begun in film No. 49 above. Contents include some review of topics from First Lesson, plus further work with signed numbers, an introduction to linear graphs, and an introduction to identities.

51. Introduction to Identities. This film consists of a small section from Second Lesson, to which appropriate titles have been added.

52. An Introduction to Linear Graphs. This film is another small section from Second Lesson, again with appropriate titles added.

v. Films Involving Class "E"

Class "E" began the study of Madison Project materials in the year 1961-1962, when they were 2nd graders. In December, 1963, they were in grade 4. They are consequently 5 years younger than the students in Class "A," 4 years younger than those in Class "B," and 3 years younger than those in Class "C."

Nametags on students in Class "E" read: "Sandy," "Nora," "Nancy," "Charlotte," "Ian," "Greg," "Barbara," "Paul," "Jeff," "Vicki," "Gary," "Ellen," "Bruce," "Brooke, "David," "Charity," "Gray," "Bob," etc.

53. A Lesson with Second Graders. This is virtually Class "E's" first experience with Madison Project material, save for a short 10 minute lesson the preceding week which

---

[1] Cf. Davis (A 1964 ab).

was made as a trial to test the readiness of this class to begin filmed lessons. The film was made during the academic year 1961-1962, when the children were in the 2nd grade. This has proved to be by far the most popular film ever made by the Madison Project, in part because of the relatively simple mathematical content, and in part because of the elegant directing of Director Robert Bleyer.

The mathematical content consists of: an introduction to signed numbers; preliminary work with variables, open sentences, and truth sets; the number line; Cartesian co-ordinates.

Excerpts from this film appear also in Education Report: the New Math, made by WNBC, Channel 4, New York City (listed here as film #9).

54. (no official title) "The Arithmetic of Signed Numbers." Class "E," as 3rd graders, academic year 1962-1963. Excerpts from this film appear in the film Creative Learning Experiences, listed earlier as film #19.

55.[1] (untitled film from videotape #VT-59) Class "E," as 5th graders, working on a lesson on identities and "list shortening" taught by Professor Katharine Kharas on March 13, 1965. The main emphasis is on generalizing by introducing new variables.

vi. Films Involving Class "F"

Nametags on students read: "Betsy," "Deidre," "Jan," "Jane," "Kathy," "Kim," etc.

56. (no official title) "Linear Equations and Monotonicity." The class considers the solution of linear equations with rational roots, using (among other methods) the monotonic property of linear functions.

vii. Films Involving Class "G"

Nametags on students read: "Michelle," "Robie," "Craig," "Dean," etc. Present records appear to indicate that in December, 1963, Class "G" were in 5th grade, but this may be in error.

---

[1] For technical details, please refer to the Quarterly Report dated March 31, 1965, p. 3.

57. (no official title) "Introduction to Functions." Class "G," academic year 1962-1963, grade 4 (but note that, as discussed above, grade level data on Class "G" may be in error). Following a suggestion of Professor W. Warwick Sawyer, of the Mathematics Department of Wesleyan University, Middletown, Connecticut, the approach goes as follows: a group of children (working together in order to reduce arithmetical errors, which would "spoil the fun of the game") make up a function. They keep their function secret, but as the class give them values of x, they respond with the appropriate value of f(x). In this way, a short table of values for the function is built up. Whenever a student can "guess" the function in words (e.g., "double the number and add 3") and can follow it up by writing the function in $\square$, $\triangle$ notation (e.g., "$(2 \times \square) + 3 = \triangle$"), this student is allowed to pick a few co-workers with whom he will make up the next function.

This lesson format can be very useful; it provides much practice in arithmetic, it introduces the concept of function, it involves properties of functions such as linearity and rapid or slow growth rates, etc., it may involve a use of graphs, and so on. In particular, it will almost inevitably come to involve the distinction between a function and what might be called a "formula," as when the students make up the rule.

$$(\square + 2) \times 3 = \triangle,$$

and some student "guesses" this function in the form

$$(\square \times 3) + 6 = \triangle.$$

Is he right? This also, obviously, involves algebraic identities and derivations.

Excerpts from this film are included in In-Service Course I, and in the film entitled Creative Learning Experiences, (#19, above).

A film made later, dealing with this same topic, is probably a more successful film. It is entitled Guessing Functions (#83, below).

58. (no official title) "Dividing Fractions." Class "G," grade 4 (but compare note above), 1962-1963. The students try to use what they already know about dividing integers in order to devise methods for dividing fractions.

59. (no official title) "Experience with Area." Class "G," probably as 4th graders, 1962-1963. By drawing figures on squared paper, the students are getting experience with various aspects of the concept of area. There also arises some question as to the set of transformations that are allowed in establishing congruence of two figures.

viii. Films Involving Class "H"

In December, 1963, Class "H" was in grade 3. Nametags include: "Kerry," etc.

60. (no official title) "Linear Equations, Monotonicity, the Number Line, Fractions, Algebraic Identities." Class "H," as 2nd graders, academic year 1962-1963. This lesson was taught by Mrs. Beryl Cochran, for the somewhat specific purpose of obtaining some film segments that were needed for work in teacher education.

ix. Films Involving Class "I"

Nametags on students read: "Lindsey," "David A.," "Bill," "Mara," "Doug," "Ricky," "Tory," etc. In December, 1963, these students were in grade 3.

61. (no official title) "Experience with Fractions." Class "I," 2nd grade, academic year 1962-1963. These children were known to have relatively little readiness for the study of fractions. In the present film, they are given some additional experience with fractions. Their response is relatively slow.

62. (no official title) "Experience with Fractions." The same children as in film #61, but approximately 5 weeks later. While their command of fractions is still relatively weak, it does show distinct improvement. The children are hampered in both films by significantly more rigidity of "mind set," or inflexibility in discarding early approximate notions in favor of more accurate later approximations to concepts, than is the case in any other Project films. Why this should be so is not known.

x. Films Involving Class "J"

Class "J" was in 5th grade in December, 1963, according to present records, which may be in error. Nametags read: "Sharon," "Lynn," "Mary," "Joel," "Karen," "Sarah," "Lawson," "Neal," "John," "Dana," "Debbie," "Cindy," "Kris," "Polly," "Skip," etc.

63. (no official title) "Arithmetic of Signed Numbers." Filmed in academic year 1962-1963, when Class "J" was apparently in grade 4. This film was made in order to obtain certain sections which were needed for work in teacher education.

xi. Films Involving Class "K"

Nametags read: "Hank," "Julia," "John," "Chris," "Susan," "Debby," "Martin," "Stephanie," "Robert," "David," "Paul," "Madeleine," "Gary," "Gordon," "Tom." In December, 1963, Class "K" was in 6th grade.

64. (no official title) "The Multiplicative Inverse of Zero." Filmed in the academic year 1962-1963, when Class "K" was in 5th grade. Various experiences in using multiplication and multiplicative inverses, designed to show that an inverse of zero would have to have very unusual properties indeed.

65. (no official title) "Experience With Angles." The concept of measuring angles as a measure of the amount of rotation; physical problems including wheels, etc. Filmed during the academic year 1962-1963, when Class "K" was in grade 5.

66. (no official title) "Classification of Functions, as Linear, Convex, or Concave," Class "K," 5th grade, 1962-1963. An experiment with a novel lesson format that was not, in fact, very successful.

xii. Films Involving Class "L"

Class "L" was in 5th grade in December, 1963. Nametags read: "Amy," "Cathy," "Pam," "Bob," "Mary," "Ellen," "Lisa," "Marshall," "Bill," "Ken," "Kathy," "Josh," "Maggie," "Bo," "Lucy."

67. (no official title) "Variables." This film was made during the academic year 1962-1963, when Class "L" was in the 4th grade. It was made for use in teacher education work, and presents the most sophisticated introduction to *variables* to be found among Madison Project materials.[1] (The concept of variable is presented elsewhere in Project materials in far more informal terms.)

68. (no official title) "The Distributive Law." This film was made in the spring of 1963, when Class "L" was in the 4th grade.

In the Madison Project approach, students do *not* study the "properties" of numbers, such as the "commutative property of addition"

$$3 + 4 = 4 + 3.$$

---

[1] Cf Robert B. Davis, *Matrices, Logic and Other Topics: A Text for Teachers.* Addison-Wesley Publishing Co., Reading Massachusetts (in preparation) Chapter One.

Instead of observing instances, the Project uses variables to write the general law, and uses logic to give this law status as an axiom.

There are several reasons for this, which the Project believes to be important. For one thing, $3 + 4 = 4 + 3$ is *not* the commutative property of addition, it is merely a true statement involving the numbers 3 and 4. Secondly, this is an excellent opportunity to gain insight into the use of *variables*, and this opportunity is much too good to be missed. Thirdly, "properties" which one expects students to rhapsodize over do not excite students very much, and rightly so, *for if these really were* merely "properties," *they would have only a part of their true value*; their value lies largely in the fact that they *imply* a great deal, and in a rather special way -- they belong on our list of axioms, from which we can derive all of our other true statements within the mathematical system.

Following this line of reasoning, the distributive law is introduced to children (in this instance at the 4th grade level), in the following way: first, the children are given a list of statements, such as

$$3 \times (2 + 5) = (3 \times 2) + (3 \times 5)$$

$$3 \times (2 + 7) = (3 \times 2) + 7$$

$$3 \times (5 + 11) = (3 \times 5) + (3 \times 11)$$

$$\forall_x,\ 5(x + 4) = (5x) + 20$$

and are asked to distinguish those statements which are *true* from those which are *false*.

After the children have done this, they are asked to try to shorten the remaining list of *true* statements by an appropriate use of *variables*. The ultimate result is, of course, the distributive law, which consequently appears in general form, with variables playing a role in the logical implicative structure with which the students quickly become familiar.

xiii. Films Involving Class "M"

In December, 1963, Class "M" was, according to present records, in grade 5. Name-tags read: "Kye," etc.

69. (no official title) "Arithmetic with Non-Standard Numerals."

The history of this film is as follows: When Class "M" were in grade 3, their teacher -- herself a student of one of the Madison Project specialist teachers -- was explaining to the class a method for performing the subtraction

$$\begin{array}{r} 64 \\ -28 \\ \hline \end{array} \; .$$

The teacher said, following a "traditional" approach: "I can't take 8 from 4, so I borrow 10 from the 60 ..."

At this point a 3rd grade boy, name Kye, interrupted: "Oh, yes you can! Four minus 8 is negative four,

$$\begin{array}{r} 64 \\ -28 \\ \hline {}^{-}4 \end{array}$$

and 60 minus 20 is 40

$$\begin{array}{r} 64 \\ -28 \\ \hline {}^{-}4 \\ 40 \end{array}$$

and so the answer

$$\begin{array}{r} 64 \\ -28 \\ \hline {}^{-}4 \\ 40 \\ \hline 36 \end{array}$$

is 36."

There is much to be said in relation to this episode, but what is perhaps most fundamental is that the teacher listened to the child's suggestion, actually thought about it, and was able to recognize that it was appropriate and was able to accept it. If creative mathematics learning is to mean anything genuine at all, it must be predicated upon such an approach, and not upon the "traditional" approach: "No, that's not the way you do it!" This matter is discussed in the Report of the Cambridge Conference,[1] in E.S.I.'s African Educational materials,[2] in the pamphlet Math Takes a New Path,[3] and in the film Education Report: The New Math (listed above as film #9).

Clearly, one would like to have on film the class in which Kye made his initial remark, and the teacher made her answer. This incident occurred, however, in a class that

---

27. [1] Goals for School Mathematics (B1963), pp. 17, 18, and especially pp. 26,

[2] Richmond, Putnam, et al. (C1963).

[3] Davis (A1963h).

was neither being filmed nor even recorded on audio-tape. This cannot be remedied subsequently and retroactively.

What was done was the following:

At the earliest possible time this class did appear before cameras, and they were asked to elaborate upon Kye's idea. In particular, they were shown the device of using negative digits, indicated by a negative sign written above the digit, as in this example:

$$\begin{array}{r} 64 \\ -28 \\ \hline 4\bar{4}. \end{array}$$

One can then turn to problems such as the following addition problem

$$\begin{array}{r} 3\bar{5} \\ +2\bar{4} \\ \hline \end{array},$$

and so on.

xiv. Films Involving Class "N"

Class "N" was taught all year, for the academic year 1963-4, by a team consisting of Professor Robert B. Davis, and Sister Francine, S. L. During this year, Class "N" were 9th graders. A detailed report of this entire 9th grade program is contained in: Robert B. Davis, Experimental Course Report/Grade Nine. Experimental Course Report #1, June, 1964, available from the Madison Project.

Nametags read: "Bev," "Marybeth," "Pat C.," "Carol," "Michele," "Pam," "Pat D.," "Susie," "Nancy F.," "Suzi," "Maureen," "Eileen," "Donna," "Kathy V.," "Kathy W.," "Cathy," "Regina," "Karen," "Chris Herbert," "Sandy," "Pat H.," "Chris Hohl," "Kathy H.," "Kathy K.," "Marian," "Clare," "Mary Catherine," "Patty," "Mary Ann," "Janice," and "Nancy O., "[1]

70. (untitled film from videotape #VT 35). Taping Session at KETC-TV, St. Louis, Missouri, Saturday, Dec. 21, 1963. There are two topics in this lesson, which runs for 59 minutes. The first topic is concerned with selection of algebraic axioms and selection of rules of logic, with the subsequent proofs of these theorems

a) $(A + B) \times (C + D) = (D + C) \times (B + A)$

---

[1] Cf. Appendices C and D of Davis (A1964g), just mentioned.

b) $2 + 2 = 4$

c) $A + (B \times C) = (C \times B) + A$

The second topic is some work with implication, contradiction, and uniqueness, based upon a sophisticated version of David Page's Hidden Numbers.

71. (untitled film from Videotape #VT 36) Taping Session at KETC-TV, St. Louis, Missouri, Saturday, Dec. 21, 1963.

There are two topics on this tape, which runs 54 minutes. The first topic continues the axiomatic algebra from Video Tape Number 35, with emphasis upon theorems involving additive inverses, including the theorem

$$^\circ(^\circ A) = A.$$

The second topic deals with truth tables, and with the mapping of the Cartesian product $\mathcal{V} \times \mathcal{V}$ into $\mathcal{V}$, where $\mathcal{V}$ is the "truth value space,"

$$\mathcal{V} = \{T, F\} .$$

72. Negative One Times Negative One Equals Positive One. Video Tape Number 37, recorded at KETC, St. Louis, Dec. 21, 1963. This lesson includes three approaches to the statement

$$^-1 \times {}^-1 = {}^+1.$$

The three approaches are:

i) use of a "model" ("postman stories")

ii) an axiomatic proof of the theorem

iii) an approach via "extension of a definition," using linear graphs, following a suggestion of Paul Rosenbloom.

73. Quadratic Equations. Recorded on May 9, 1964, at KETC-TV, St. Louis, Missouri. A Lesson on the derivation of the quadratic formula. (Videotape #VT-43).

74. Introduction to Infinite Sequences. Recorded on May 9, 1964, at KETC-TV, St. Louis, Missouri, on Videotape #VT-44.

This is the first time these students encounter this topic. It arises out of two monotonic sequences related to $\sqrt{2}$, which in turn arises out of the attempt to achieve a general quadratic formula.

75. What is Convergence? Recorded May 23, 1964, at KETC-TV, St. Louis, Missouri. This lesson continues the work on infinite sequences which was begun on VT #44. The approach is somewhat similar to the "What's My Rule?" approach to functions: the teacher associates numbers with various sequences, or else declines to do so (saying "that sequence is divergent"), and the students are asked to describe, as precisely as they can, the procedure that the teacher is using. (VT #46)

76. Bounded Monotonic Sequences. Recorded May 23, 1964, at KETC-TV, St. Louis, Missouri. This lesson continues the work on sequences from Video-Tapes VT #44 and VT #46. Two of the students decide that every bounded monotonic sequence converges, and another student decides that, since it seems unlikely that this could ever be proved from CLA, DL, etc., it needs to be added as an additional axiom. (VT #47)

77. Introduction to the Complex Plane. Recorded May 23, 1964, at KETC-TV, St. Louis, Missouri, on Video Tape VT #48. In the lesson shown on this film, matrix names are given to points of E_2.

xv. Films Involving Class "O"

Class "O" was in kindergarten at the Hilltop School, in Ladue, Missouri, during the academic year 1963-4. They were taught mathematics all year long by a team consisting of Mrs. Marilyn Ward (their regular classroom teacher) and Mrs. Doris Machtinger (a "visiting specialist teacher" for mathematics).

A complete report of this work is given in

> Doris Diamont Machtinger, Experimental Course Report/Kindergarten. (Report No. 2, July, 1965). (in preparation).

Nametags read: "Kim B.," "Mark," "David O.," "Karen," "Lauren," "Terry C.," "Leslie."

78. (untitled film, from Video Tape VT #34, 1st part) Recorded on December 21, 1963 at KETC-TV, St. Louis, Missouri.

Various experiences in ordering and the properties of integers. (e.g. Kim's rule says that "each number is one larger than the one before it.") Children order themselves by comparison of their heights. Number line work establishes that there are numbers the same distance to the left of zero as there are to the right. These are called Terry's numbers (those to the left of zero) and David's numbers (those to the right of zero). Teacher mentions that many people call these positive and negative numbers.

79. (untitled film, from Video Tape VT #34, 2nd part) Recorded on December 21, 1963 at KETC-TV, St. Louis, Missouri.

Experiences which deal with grouping. Children establish symbols for five (⁙) etc., discuss need for re-use of symbols. Class also works on counting of groups of pebbles by using a 1 - 1 correspondence between fingers and pebbles; putting 10 pebbles in a bag and telling how many "bags" and "loose ones" in the group.

xvi. Films Involving Class "P"

Nametags read: "Austin," "Margo," "Frank," "Collis," "Sara," "Dana."

80. (untitled film, from Video Tape VT #38) "Subtraction and Division." This lesson was taught by Beryl S. Cochran on January 11, 1964, in West Hartford, Connecticut. At that time, these children were in grade 2, according to our records.

Concrete use of beans and counting board to answer problems of form $15 - \square = 8$. Some beans are counted into one of the depressions in the board and a child removes part of them. Several children count those that are left and "predict" how many the first child removed. The problem is then extended to problems of the form $\frac{12}{3} = \square$. Some beans are counted into a depression in the board, then a child is asked to count them into piles of 3 (or some other number). The children are asked to predict the number of piles. Finally, this is extended into an abstract problem by asking what would happen if there were $\frac{1}{2}$ a bean in each pile. When the problem $\frac{3}{\frac{1}{4}}$ is asked, the children are hesitant until the problem $\frac{3}{\frac{1}{2}}$ is asked. When this problem has been solved, the problem $\frac{3}{\frac{1}{4}}$ is correctly answered.

xvii. Films Involving Class "Q"

Nametags read: "Yvonne," "Janice," "Stuart," "Dennis," "Alesia," "Denassa."

81. Graphs and Truth Sets. This lesson was recorded on May 9, 1964, at KETC-TV, at St. Louis, Missouri. At that time, Class "Q" were second and third graders, described both as "culturally-deprived" and "school problems." The teacher is Mrs. Katherine

Vaughn, who became familiar with Madison Project materials by attending sessions that used In-Service Course I.

The contents of the lesson are: signed numbers, the number line, Cartesian coordinates, open sentences and truth sets, the correct use of variables, and linear graphs.

xviii. Films Involving Class "R"

Nametags read: "Earline," "Joyce," "Ralph," "Robert D.," "Gloria," "Helen."

82. Postman Stories. This lesson was taught, and videotaped, at KETC-TV, St. Louis, Missouri, on May 9, 1964. At that time, these children -- many of whom are considered "culturally-deprived" -- were in grades 6 and 7. The teacher is Robert B. Davis.

Observation of the class a week or so earlier had suggested that they were very confused over "Postman Stories" -- a story-type model which the Madison Project used to create a symbolism for the arithmetic of signed numbers and which often gives teachers trouble. Consequently, the present film was conceived as a lesson that would help to show teachers how to straighten out confusions regarding the proper use of "Postman Stories."

Perhaps unfortunately, by the time it was possible to get the children in front of cameras, their confusion about "Postman Stories" had straightened itself out -- even though their teacher had avoided this topic completely in the interim. (This lends further support to those who see "cognitive dissonance" or "mastery drives" as essential components in learning.)

The teacher on the filmed lesson was therefore compelled to change plans in mid-lesson, and to switch to a problem which would involve using "Postman Stories." The problem chosen was to graph the truth set of

$$(\square \times \square) \quad + \quad (\triangle \times \triangle) \ = \ 25.$$

In viewing the filmed lesson, one can observe marked student growth: at first the children do all of the problems on the arithmetic of signed numbers incorrectly. As they make up the corresponding postman stories, however, they correct their work without appealing to the teacher. Finally, they begin to see enough patterns to be able to handle the arithmetic of signed numbers without recourse to postman stories.(Video Tape VT #45)

83. Guessing Functions Recorded at KETC on May 23, 1964. This is one of the Madison Project's most successful films. It follows the standard "guessing functions" lesson format (cf. film #57 above). (Video Tape VT #49) (The children were in grades 6 and

7 when this lesson was recorded.)

84. Units for Measurement -- Distance and Angles.

Recorded at KETC, May 23, 1964. This lesson was intended to give children experience in measuring angles and distances, with appropriate units in each case. The lesson was unsuccessful. (Video-Tape VT #49) (The children were in grades 6 and 7 when this lesson was videotaped.)

xix. Films Involving Class "S"

85. Videotape VT #50. No kinescope has ever been made, due to the poor technical quality of VT #50. (Lesson taught by Katharine Kharas, Gainesville, Florida, Oct..5, 1964.)

86. Introductory lesson on "Postman Stories," taught by Robert B. Davis in Gainesville, Florida, on October 5, 1964. Recorded on Video Tape VT #51. The lesson is unsatisfactory; the students are not in the mood for serious work and serious thinking.

xx. Films Involving Class "T"

This class was in grade 9 during the academic year 1964-5 at Nerinx High School, Webster Groves, Missouri. They were taught all year long by a team consisting of Sister Francine, S. L., and Professor Robert B. Davis. The year's work was generally similar to that of class "N," the previous year, but differed primarily in two respects:

i) The program for Class "T" included considerably more logic -- indeed, perhaps too much emphasis on logic;

ii) The lessons for Class "T" made more use of "small-group" or "committee" arrangements within the classroom, individualized instruction, and physical apparatus.

Nametags read: "Mary W.," "Cathy C.," "Debby," "Kathy," "Ellen," "Ruth," etc.

87. (recorded at KETC on November 28, 1964) A Language for Algebra. Logic used as a language for the discussion of axiomatic algebra. (Video Tape VT #53)

88. Algebraic Systems -- Intuition vs. Formal Description (recorded at KETC, St. Louis; Video Tape VT #55)

Class established the following axioms for any set S.

1. $S \neq \emptyset$
2. $\forall_x \; \forall_y \; (x \in S) \wedge (y \in S) \Longrightarrow x * y \in S$
3. $\exists I$ such that $\forall_x \; x * I = x$

They then decided that several sets S, agreed with these axioms -- e.g., the real numbers work if

1. $* = \times, \quad I = 1$
2. $* = +, \quad I = 0$

and as another example, 2 x 2 matrices work if

1. $* = + \quad I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
2. $* = \times \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Class was then able to deal with the following new system

There are two operations, * and +
The system has closure
The system is commutative
Each of the operations * and + is distributive over the other.

They tried this with sets of names, but time prevented final conclusion that * was the intersection and + was the union of sets.

xxi. Films Involving Class "U"

89. Physical Apparatus in the Mathematics Classroom -- The Tower Puzzle. A class of mainly culturally-deprived children, from grade 7 in a lesson taught by Donald Cohen and Ogie Wilkerson, on April 3, 1965.

Nametags read: "Arlene," "Vernel," "Oliver," "Tommy," "Andrea," "Augusta," "Stephansen."

The main point of this lesson was to record the lively and interesting mathematics classes that Donald Cohen regularly teaches, using small-group "committees" and an abundance of physical apparatus. The actual lesson lasts all morning, and is in fact too long; moreover, it is built upon inadequate previous readiness, particularly in regard to the way that the students do not feel at home with the notation for variables, nor with exponential functions.

In the famous "Tower Puzzle," if $\square$ denotes the number of discs, and $\triangle$ the minimum number of moves required to solve the puzzle, then

$$2^{\square} - 1 = \triangle .$$

From another point of view, if P_n represents the minimum number of moves required to solve the n-disc puzzle, then one has the recurrence formula

$$P_n = (2 \times P_{n-1}) + 1.$$

(This is the only film ever made, thus far, by the Madison Project that was not recorded first on video tape. Instead, this film was recorded directly with 16-mm. film cameras, which seemed appropriate for the "small-group" or "three-ring-circus" type of lesson.

xxii. Films Involving Class "V"

These 5 half-hour lessons were taught at San Diego State College, California, during June, 1965, at a one-week workshop organized by Dr. John Huffman. The teacher is Robert B. Davis. This class is the first truly racially integrated class that the Project has had the privilege of filming. Nametags read: "David," "Ron," "Linda," "Barbara," "Gerald," "Billy."

If further evidence to support careful readiness building were needed -- and it is not-- these lessons would provide it. For various reasons, partly related to a desire to show the teacher-participants as much as possible during the one-week workshop, and partly related to the demonstration teacher's error in misjudging the students, he moves forward too quickly, without all the necessary preparation in readiness building. As a result, the lessons do not progress satisfactorily -- as nearly always happens, haste makes waste, and the end result is slower progress than would have been obtained with more careful readiness building.

90. A Week of Mathematical Discovery.

I. Monday. The contents of the lesson are

1) an introduction to signed numbers via Pebbles-in-the-Bag,

2) Open Sentences and the rule for substitution;
3) an introduction to quadratic equations;
4) plotting points on Cartesian Co-ordinates via the game of Tic-Tac-Toe. The game was played entirely in the 1st quadrant.

91. A Week of Mathematical Discovery.

II. Tuesday. This lesson consisted of

1) the arithmetic of signed numbers via the "Postman Story" model described in #82.
2) a brief introduction to the "Tower Puzzle," described in #89.
3) some work with functions via the game "Guessing Functions" described in #57.

92. A Week of Mathematical Discovery.

III. Wednesday. This lesson was largely a continuation of the one described in #91 above. The class continued with guessing functions, discussed some of their discoveries about the Tower Puzzle and continued their investigation of Cartesian Coordinates by playing Tic-Tac-Toe in four quadrants.

93. A Week of Mathematical Discovery.

IV. Thursday. This lesson was intended to clarify the group's varied notions about the arithmetic of signed numbers. Using the "Postman Story" model, they dealt with several problems involving the operations of addition, subtraction and multiplication with signed numbers. They then continued with some linear graphing, a topic they were now prepared for, and completed the lesson by playing Tic-Tac-Toe in four quadrants.

94. A Week of Mathematical Discovery.

V. Friday. Friday's lesson was intended to exhibit a culmination of the week's learning, and to some extent it does. The unfortunate effects of too much haste and too little careful readiness building all week are, however, evident. The lesson is concerned with the classical tower puzzle which was studied also in the lesson on film #89. The pay-off of the lesson, as taught here, lies in the question "how many moves would be required to solve the puzzle using 100 discs?" Had there been more time -- and, perhaps, had the children been slightly more ready for it -- an even better culmination is to estimate how long it would take to solve the 100-disc problem. The answer is staggering.

IX. SOME FAILURES

It has been charged that curriculum projects do not report their failures. Actually, normally-encountered failures are of several kinds, including:

i) Temporary failures, or "individual instances of failure," where the teacher (or someone else) does something wrong on a tactical level. These might be likened to driver-error automobile accidents. Failures of this type occur every day, to everybody. Somebody made a decision that was less than optimal, and the results turned out to be less than optimal. Such errors become interesting mainly in those cases where some sort of persistent pattern can be detected. In such cases, one gets in effect a further contribution to a growing "theory of instruction."

ii) Failures that appear to be absolute in nature, where the conclusion is: "Don't ever try it! There is no good way to do it!"

iii) Somewhat intermediate between accidental "personal" errors (case i) and "inevitable laws of nature" (case ii), are those failures that occur when no one has been able to succeed, but it looks as if success should be possible if you could find the right way of going about things.

The many "personal" errors of type i which the Madison Project experiences add to the Project's growing "theory of instruction";[1] for example, "be sure to build adequate readiness in advance of the time when you are going to need it, "use a specific illustration whenever possible instead of a general abstract discussion," and so forth.

The most tantalizing pieces of unfinished business, by way of "type iii" failures, have been the Project's persistent failure to get fully satisfactory results with 7th grade children, and the Project's inability to determine why certain classes of children work together well year after year, whereas other classes never get off the ground.

The main "type ii" failures the Project has encountered are two in number: its unsuccessful effort (in 1961 - 1962) to teach linear algebra to sixth graders who had had one previous year of experience with Madison Project materials, and its unsuccessful effort (in 1963 - 1964) to teach the theory of limits of infinite sequences to 8th graders. Actually, these failures have perhaps been misrepresented, for in both cases (and especially with the 8th graders) success seemed to be just around the corner, and there is a strong suggestion that both topics (or at least the work on limits for 8th graders) could be taught successfully at these grade levels if someone could find the proper way to go about doing it.

---

[1] Cf. Davis (A1964c).

Linear algebra for grade six may, in fact, be premature; the topic may be better left until the student is at least a few years older.

In both cases (the 6th graders and the 8th graders) the children involved do not seem to have suffered; indeed, they appear to have gained a good deal, but not as much specific subject matter mastery as was intended.

In connection with the claim that these children did not seem to have suffered from the exploratory teaching, it is worth recalling the Project's unremitting efforts to spare children from feelings of embarrassment or inadequacy, the Project's use of "the light touch" in the art of teaching, the Project's consistent policy of not expressing psychologically negative judgements on a child's work, the Project's provision of alternatives for children to work on when they wish to withdraw from a central activity, and the fact that the Project materials are supplementary to a pre-existing complete mathematics program in each school.

X. COLLECTING AND INTERPRETING DATA; USE OF FEEDBACK; EVALUATION

Curriculum evolution projects are concerned, on the one hand, with the evolutionary change of a man-made institution, and on the other hand with the process by which children learn and grow . The first of these concerns involves an historical process; the second, a natural phenomenon that can in part be related to laboratory studies, and in part to naturalistic observation of growth and development. Hence the concerns of a curriculum evolution project are not entirely "experimental," nor entirely "naturalistic," nor yet entirely "historical." Assessing the impact of a curriculum evolution project consequently requires a broad methodology.

In the case of the Madison Project, the "methods and materials of the Project" consist of fully-worked-out flexible classroom lessons, which teachers learn by viewing films or by observing a live demonstration class, and which they subsequently teach to their own students. Consequently, from an experimental point of view the "independent variables" are the measures of the extent to which a student has in fact actually experienced classroom lessons of this sort. The "dependent variables" are descriptions of student behavior that describe the results of students experiencing such lessons.

Keeping this in mind, we shall now consider questions that may reasonably be asked about the effectiveness of the Project's "methods and materials," and about the effectiveness of the Project itself.

(It is also important to keep in mind that the Madison Project program provides a supplementary "enrichment" program of an ungraded nature, which is actually most relevant in grades 3 through 9, or thereabouts. Except for recent exploratory work in kindergarten, grade 1, and grade 9, the Project has not constructed a total mathematics program for each year's work.)

1. The Description Problem -- What is "The Madison Project" Program?

As mentioned above, the "methods and materials of the Madison Project" (which we shall henceforth abbreviate to "the authentic program" or "AP") consist of a sequence of fully-worked-out flexible classroom lessons which are recorded on film, which teachers learn either by viewing these films or by observing a live demonstration class (with subsequent discussion and study), and which teachers thereafter present to their own students, trying (for the most part) to adhere reasonably closely to the lessons on the films, while at the same time utilizing student initiative and student decision-making as fully as possible.

Studying such a program, then, can proceed as follows:

i) We can define "the authentic program" or "AP," as consisting of the sequence of lessons recorded on film, plus an extrapolation of the dynamics for adjusting to varying student leads and needs, as shown on the films.

ii) We can try to identify some crucial attributes of these lessons (including the "adjustment dynamics" just mentioned), and, wherever appropriate, attempt even to quantify them. We shall call this the "description problem."

iii) We can use various methods for inquiring into the appropriateness or value of the AP. We shall call this the "optimization problem."

iv) With regard to the diffusion of Madison Project ideas throughout various school systems, small and large, we can inquire into the extent to which children really are experiencing the AP, as opposed to experiencing lessons quite different from the AP. We shall call this the "authenticity problem." As part of this same question, we can include the question of how the diffused approximate programs (which we shall call "Approx P") differ from AP.

A. With regard to the "description problem" -- i.e., the identification of crucial attributes of the AP, including the dynamics by which the teacher adjusts to student suggestions, etc. -- the Madison Project has relied primarily upon the opinions of consultants who have viewed films, upon extensive discussion among Madison Project teachers, and upon viewing quite different non-Madison Project classes as a source of "contrast cases." One publication reporting this has appeared, namely

> Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, Vol. 2, 1964, pp. 146 - 162.

B. Concerning the "optimization problem" -- i.e., how can the AP be improved -- it is clear that all of the efforts described under A, above, can be directed also toward this end, and they are.

C. Concerning the "authenticity problem" -- i.e., how does the program as it operates in schools differ from the AP, the Project has attacked this problem with the following methods:

i) Tape recording actual classroom lessons from various schools across the nation, duplicating these tapes, and submitting them to a panel of 61 professional people, including mathematicians, logicians, physical and biological scientists, secondary school mathematics teachers, elementary school teachers, principals and school supervisory personnel, guidance counsellors, clinical psychologists, psychiatrists, psychoanalysts, specialists in motivation, social psychologists, specialists in cognition, and others. (Cf. Davis A 1963a.)

ii) Maintaining an extensive program of observation of teaching in various schools, the observers being experienced Madison Project teachers.

iii) In a few cases, teachers in various schools who are relatively remote from the Project have been selected to teach lessons in front of TV cameras, and video-tapes and films have been made. (These teachers include Elizabeth Bjork, Lincoln, Massachusetts; and Katherine Vaughn, St. Louis Public Schools.) These films are then viewed by the normal broad range of viewers, as will be described below.

D. Some conclusions concerning the "authenticity problem." The results of the kind of studies just described are partly encouraging, but only partly so.

i) Some teachers in schools around the nation teach a very authentic version of the AP, which is successful in their own eyes, and which the Project also considers successful.

ii) In far too many cases, the authenticity of the program is marred by mathematical errors, on even quite elementary questions such as whether (5,5) is a legal replacement for the variables (x,y) in

$$x + y = 10,$$

on how to plot points in Cartesian coordinates, etc. This occurs even though the Project has taken some pains, in its explanatory printed materials, to make these matters as clear as possible.

iii) Teachers fall in a bi-model distribution with regard to the extent of student-centeredness in their teaching;

iv) Teachers who attempt to maintain a student-centered approach (i.e., listening carefully to student suggestions, etc.) often lose sight of the mathematics in the process; many teachers find it hard to help the student to discover reality: they either keep a version of reality in mind, forget the student, and lecture to him; or else they keep the student in mind, and engage in a dialogue with him, but the dialogue never gets to the point of clarifying the mathematics. Evidently, elementary teachers need more experience in discussing mathematical ideas.

v) A particular difficulty, encountered by virtually every new curriculum Project, has been named " lysophobia" by Professor Robert Karplus, of the Berkeley, California Elementary School Science Curriculum Improvement Study. Whereas science and creative mathematics are essentially and necessarily tentative, uncertain, and open-ended, the traditions of elementary school teaching in many instances are authoritative, definite, absolute, and certain. Schwab, for example, reports[1] that he was discouraged by a publisher from reporting that the characteristic number of human chromosome pairs was uncertain. The publisher claimed that teachers would reject a book that failed to give a definite number in answer to such a question. This same phenomenon has been encountered by J. Richard Suchman in his work on inquiry training, and by Charles Walcott in elementary school biology.

---

[1] Schwab (C1964), p. 62.

It should be emphasized that the difficulty here is with some teachers, not with children. Children know that they live with incompleteness and uncertainty; scientists know that no other state is available to living human beings. Unfortunately, teachers have all too often been taught that every question has exactly one right answer, and that the child is entitled to know what it is -- or, perhaps, should even be required to know what it is.

This difficulty becomes especially acute the moment that measurement of any kind becomes involved, as it must in any broad mathematics or science program. Some teachers have rejected the Project's experiments with linear elasticity using weights suspended on a spring, because of the fact that they could not get "right" answers for the amount the spring stretched. The Project has consequently modified its materials so as to come at this problem from two sides: first, Mr. Donald Cohen has introduced a sequence of empirically-determined functions that are determined by counting, rather than by measuring (e.g., how many moves are required to solve the tower puzzle with n discs?); second, the Project has developed lessons dealing specifically with the problem of measurement error and the fact that, in a measurement problem, there is no single exact answer (Cf. the film Average and Variance). After *these two preparatory steps have been taken,* it is hoped that it will then be possible for teachers to work with empirically-determined functions where measurement is involved.

vi) Perhaps more important than monitoring authenticity (particularly in regard to the least controvertible aspect, namely actual mathematical errors in teacher presentations) is the question of what does the Project do to try to improve teacher performance? Some of its actions are: developing teacher education programs with more depth (e.g., no mathematical errors have ever been detected in elementary school teaching by seniors or graduates from the Webster College elementary school specialist teacher program; this program provides adequate depth -- with 30 credits of college-level mathematics -- to avoid such errors); the use of two teachers working as a team with the same group of students; significant expansion of the usual extent of observing and assisting classroom teachers; operation of a continuing program of conferences, seminars, and workshops for elementary classroom teachers; use of advanced Webster College undergraduates as technical advisors for mathematics, to experienced elementary school classroom teachers (an arrangement which has worked surprisingly well, and to the mutual advantage of both groups); frequent review of Project materials to try to avoid teacher errors, by clarifying exposition where needed, etc.; separation of Project materials into relatively "stable" material that many teachers can teach well, versus "exploratory" or "advanced" material that most teachers should not attempt; allowing teachers across the nation to tape-record one of their classes and send the tapes to the Project for comment.

2. Does The Madison Project Produce Change?

What evidence is there that the Madison Project is successful in producing curriculum evolution?

A. Examination of various programs. The Madison Project has participated in the writing of the important African Mathematics Program, produced at Entebbe, Uganda, under the leadership of E.S.I.; several commercial textbook series appear in some cases to deal similarly with similar topics, as compared to the AP; the important E.S.I. Pre-College Center Program explicitly makes use of some Madison Project materials; New York City and Chicago acknowledge Madison Project influence on their elementary school curricula; the Colorado State Department of Education has prepared an In-Service Course for elementary teachers which shows (and acknowledges) important Madison Project influence; the Brandeis University 1965 summer program for the education of Peace Corps volunteers showed (and acknowledged) Madison Project influence; the Madison Project is, of course, explicitly involved in large programs in Chicago, St. Louis, Washington, D. C., San Diego County, and New York City; Madison Project materials are used rather extensively in several schools in England, and also in Jobs Corps programs in the United States.

B. Invitations to participate in college programs, school workshops, and professional meetings. The Project has been invited to participate, and has participated, in the following programs: NSF Institutes at University of Arizona, Tulane University, University of Vermont, Syracuse University, Webster College, Harvard University, Yale University, University of Michigan, University of Illinois, Washington University, University of Maryland, Earlham College, Colgate University, Hamilton College, Gallaudet College, Purdue University, East Tennessee State University, Miami University (Oxford, Ohio), Bowling Green State University (Bowling Green, Ohio); college programs at St. John's University (Long Island), College of Idaho, University of New Hampshire, Danbury (Connecticut) State College, William Jewel College, Humboldt State College (Arcata, California), University of Florida (Gainesville), Stillman College (Tuscaloosa, Alabama), Indiana State (Terre Haute), SUNY at Oswego, SUNY at Oneonta, SUNY at Stony Brook, Harper College (Endicott, New York), Henderson State Teacher's College (Arkadelphia, Arkansas), Salem (Massachusetts) State College, University of Kansas City, Chicago Teacher's College, Columbia University, San Diego State College, University of Missouri (Columbia, Missouri), Antioch, Hunter College (New York City), Knoxville College (Knoxville, Tennessee), Plymouth State College (Plymouth, New Hampshire), Harris Teacher's College (St. Louis, Missouri), Northern Michigan University (Marquette, Michigan), Lindenwood College (St. Charles, Missouri), Southern Connecticut State College (New Haven), University of Pittsburgh (Pittsburgh, Pennsylvania), Illinois State Normal University (Bloomington, Illinois), Carleton College (Northfield, Minnesota); meetings and workshops (in addition to those listed above) in California, Washington, Oregon, Colorado, Utah, Arizona, Minnesota, Wisconsin, Michigan, Oklahoma, Texas, Alabama, Florida, Missouri, Kansas, Virginia, Tennessee, Washington, D. C., Pennsylvania, New York, Connecticut, New Jersey, Indiana, Ohio, Illinois, Massachusetts, Canada, England, and both coasts of Africa; and invited presentations at Educational Testing Service, and at various meetings (at M.I.T.'s Endicott House, and elsewhere) arranged by Educational Services Incorporated. (These E.S.I. meetings serve an important evaluation function, and will be discussed at more length below.)

C. Mention of Madison Project methods and materials in reports, reviews, etc. A large number of articles, reports, reviews, and other publications mention the Madison Project, sometimes discussing it at considerable length. Cf. the Bibliography at the end of this report.

D. Frequent mention in the daily press. It has been impossible to keep track of the references to local usage of Madison Project methods and materials that have appeared in the daily press, from Arcata, California to Fort Lauderdale, Florida, but these accounts have been extremely numerous, and have included newspaper accounts in New Haven, Connecticut; Weston, Connecticut; New York City; San Diego; St. Louis; Chicago; Amherst, Massachusetts; Boston, Massachusetts; Syracuse, New York; Hartford, Connecticut; Los Angeles, California; Tuscaloosa, Alabama; Dover, Delaware; and many other cities and states.

E. Television programs involving the Madison Project have been broadcast by KMOX-TV, St. Louis; WNBC (Channel 4), New York City; National Educational Television (NET) in 85 cities; and by additional commercial stations in other cities.

F. In an attempt to keep track of the evolving curriculum, the Madison Project administers a questionnaire to a large number of teachers, primarily in the midwest and the northwest. The purpose of the questionnaire is to record teachers' changing perceptions of the school mathematics program over a span of years.

G. Teachers attending professional meetings and workshops show clear evidence of a growing awareness of Madison Project methods and materials.

H. Attendance at Madison Project sessions at professional meetings is large, and has been growing.

I. The Project follows up teachers who have attended programs at Syracuse University and at Webster College. A reasonable number of these report use of Project methods and materials in their own schools.

J. In the San Diego workshop, June, 1965, 234 teachers attended for an entire week, even though no credit was offered. Indeed, attendance grew toward the end of the week.

K. The Project has just received approval from its sponsors to distribute, and has distributed 57,500 copies of Newsletter No. 1, which announces the first general public availability of Madison Project films. Five titles have been released for general distribution:

A Lesson With Second Graders
First Lesson
Second Lesson
Graphing a Parabola
Guessing Functions.

Since the mailing of Newsletter No. 1 has just occurred, it is too soon to say what the response will be. Prior to the mailing of Newsletter No. 1, Project inquiries arrived at about the rate of eight letters per business day, or about 2,000 inquiries per year. (This, of course, means reaching far more than 2,000 teachers per year, since many inquiries came from supervisors, coordinators, or college faculty working with methods courses, etc.) How this rate will be affected by the mailing of Newsletter No. 1 remains to be seen.

3. Are These Changes Desirable?

Evidently one question that must be answered -- at least as well as we are able to answer it -- is: are these changes that the Madison Project attempts to make desirable? In this section we shall consider eight methods by which the Project attempts to answer this question. (In section 5 we shall look at this same question from the opposite point of view: what undesirable consequences may be involved?)

i) Evidently, the "tape panel" -- the tape recording of actual classroom lessons, and the study of these tapes by a panel of 61 professionals from various areas of specialization — which is aimed primarily at description, and secondarily at authenticity, must automatically make an important contribution also to the optimization problem, and has done so.

ii) Clearly, much of the "value" question is -- at least for the present -- non-empirical, and can best be pursued by comparing the objectives of different programs, and arguing about rationales. Such discussion is a continuing activity among Project personnel.

iii) Again, the reports received from those on the firing line are important. One criterion, then, is: what do experienced Madison Project teachers report about the "feel" of the lessons? Lessons are ordinarily modified until a firm consensus exists in their favor.

iv) Presentation at discriminating forums. An approach which the Project values especially highly has been to make extended presentations at various important forums -- especially at E.S.I. meetings at Endicott House, at Professor Bruner's "Learning about Learning" sessions, and at meetings called by the Social Science Research Council.[1] At such meetings it has been possible to view films of classroom lessons, discuss problems, doubts, and future plans, and get significant responses from such people as Jerrold Zacharias, Francis Friedman †, Gilbert Findlay †, Robert Karplus, Myron Adkins, David Page, Patrick Suppes, W.T. Martin, Andrew Gleason, Philip Morrison, David Hawkins, E.E. Moise, Stephen White, Jerome Bruner, Jerome Kagan, Earl Loman, Charles Walcott, Miriam Goldberger, Gardner Quarton, Leonard Sealey, and others.

These meetings are of the highest possible value, especially when they are held over an extended period of time at Endicott House, or in a similar location, for at least three reasons: the participants are very well chosen and speak with a maximum of wisdom and good judgement; the participants attend the meetings for a long enough period of time to become significantly involved in the questions at hand; the diversity of the participants (physicists, mathematicians, logicians, teachers, educators, biologists, psychologists, psychoanalysts, etc.) creates a broad point of view that focusses on the problem at hand, and not on the currently-popular methodologies within a single discipline.

v) Opinions of consultants. The Project has a list of distinguished consultants who render opinions on lessons. These, too, are of the greatest value, and do result in modifications of Project materials.

vi) The Project checks the internal consistency of its pedagogical rationale against relevant laboratory results, against systematic classroom studies, and against currently popular models for human behavior. Cf., for example, Kersh (C1964), Evan and Zelditch (C1961), Flanders (C1959), Hopkins (A1965), Harlow (C1932), Day and Hamblin (C1964), Wittrock (D1964), Snygg (C1963), Gilchrist and Snygg (C1965), Berne (C1964), Gold (A1965), Cleary (A1965), Tolman (C1958), Torrance (C1963), Rogers (C1959), Ashton-Warner (C1963), Dienes (C1964), Courtis (C1963), Cantor (C1961), Erikson (D1959), Smedslund (C1964), Hunt (C1964), Vygotsky (D1962), Henry (C1963), Sawyer (C1964), Lewin,(C1942), Combs, Kelley, Rogers and Maslow (C1962), Harlow, Harlow, and Meyer (C1950), de Charms, Carpenter, and Kuperman, "The 'Origin-Pawn' Variable in Person Perception," Sociometry (to appear), and others.

---

[1] Cf. Social Science Research Council (B1965), p. 10.

† Deceased

vii) The Project checks the internal consistency of its mathematics, in the sense of notation, sequential order of topics, etc. In the same way, the Project checks its materials for sufficient diversity in points of view and applications.

viii) Data collected by schools using project materials. Every school with which the Project works has its own usual procedures for assessing student growth and achievement. In every case that has been reported to the Project by a participating school, the addition of Madison Project materials as a supplementary enrichment program appears to have improved student performance as judged by the school's usual assessment procedures. This includes not merely scores on standardized achievement tests, but also the judgement of guidance counsellors concerning student attitudes, records on absenteeism, and the opinions of teachers concerning "carry over" into student performance in areas other than mathematics. It even includes judgement of administrators concerning the enhancement of teacher effectiveness in mathematics, and here again there is a reported "carry-over" gain in teacher effectiveness in areas other than mathematics.

ix) The Cleary Report. A Madison Project Study (Cleary A 1965), conducted by Robert Cleary of Educational Testing Service, Inc., compared three 7th grade classes. Class A was a Madison Project class that had been studying Madison Project materials since third grade; Class B was a Madison Project Class that had one year of experience with Project materials; Class C was from a third community in which Madison Project materials were not used, although the mathematics program in the school was of very high quality and would ordinarily be considered "modern." The comparison was based on 45 test items made available by Dr. Roger Lennon, Director of the Test Division of Harcourt, Brace and World, Inc., from among those used in preliminary trials for the mathematics test of the Stanford Achievement High School Test Battery. World Book had run trials on these items in various kinds of schools, and had item statistics relating to performance in their list of "lighthouse" schools -- i.e., well-financed schools in upper middle class or upper class suburban or exurban communities where the schools are ordinarily regarded as among the strongest in the nation. All three classes in the Cleary report came from such schools and such communities; all three classes were judged to have approximately equal socio-economic and cultural backgrounds, the schools had comparable staff and facilities, and there were no significant differences in intelligence test scores.

The test items dealt with linear and quadratic equations, matrix algebra, axiomatic algebra, simultaneous equations, graphs of functions on Cartesian coordinates, and similar topics.

The test result showed a difference favoring Class A over Class C that was significant at the .05 level, but performance of Class B was intermediate between the performances of Class A and Class C, and was not different from either A

> or C at a statistically significant level. This confirmed a priori estimates of experienced observers. All three classes performed better than the item statistics for the national "lighthouse" norms for grade nine. Indeed, no good observer could have doubted that all three classes were very good at mathematics, and Class A was the best of all. (Actually, Class A's greatest seeming advantage, originality, inventiveness, and creativity, was essentially not called into play by these 45 items.)
>
> It may be interesting to note that the "control" school system that provided Class C for the Cleary report has subsequently chosen to become themselves one of the schools participating in the Madison Project program.

4. Diffusion and Quality Control.

What happens to Madison Project methods and materials as they diffuse outward from those schools and teachers most experienced in working with them, and begin to be used by teachers and schools that have had no direct contact with the Madison Project or any of its workshops and courses?

This point was discussed at some length, under the name of the "authenticity problem," in Section 1 of Part X of this report. That discussion will not be repeated here, but one remark needs to be added. Sociologists have long distinguished between the "formalities" and the "informalities" of social situations.

A parallel distinction is important in the collection of educational data; the Madison Project procedure includes introducing observers not known to be involved either with education or with the Project, in order that these observers may participate in conversations with teachers, parents and students. Such observation is extremely revealing. In some cases the reports have been gratifying, in others they have not been. In instances where such informally-collected evidence suggests that the Project materials are not working out well, the Project usually attempts to discourage further use of these materials, or else to encourage further study by the teachers involved.

The Project's informal observers have frequently encountered other experiments in progress; it is worth noting that informal reports do not always confirm the more formal reports that are submitted via questionnaires, formal written reports, etc. If concern is with what really happens in school and especially with people's actual attitudes, then informal channels of reporting are essential.

Informal data is kept confidential, and is used -- if at all -- with both tact and discretion.

Another aspect of "quality control" data collection will be considered in Section 6, below.

5. Avoidance of Harmful Outcomes.

What undesirable outcomes result from the introduction of Madison Project methods and materials? The Project's methods for answering this question include:

i) The tape recording of actual classroom lessons, as described above, with the tapes being analyzed by a group of professional consultants, including mathematicians, logicians, physicists, physical and biological scientists, elementary teachers, secondary mathematics teachers, school administrative and supervisory personnel, clinical psychologists, psychiatrists, psychoanalysts, school mental health specialists, social psychologists, motivation specialists, specialists in cognition, and others.

ii) Systematic tape-recorded interviews with students and teachers, conducted by an experienced clinical psychologist.

iii) Direct observation of many classroom lessons by a variety of observers;

iv) Presentation of films (or teaching of live demonstration classes) at discussion sessions of professional meetings;

v) Presentations at E.S.I., etc., as discussed above;

vi) Study of Project methods and materials by consultants;

vii) Following (in some cases) the same children for as long as six consecutive years.

viii) Use of informal observers, as discussed in Section 4 immediately above.

ix) Checking ordinary school records on performance, growth, absenteeism, etc., as discussed in Section 3 above.

There is, at present, no evidence of undesirable outcomes beyond one or two which are inevitable:

i) Use of any new curriculum involves decisions, and any issue within a school which involves decisions creates the possibility of disagreement. The question of coping with decisions and disagreement is a continuing question for every school system, and must be dealt with in a satisfactory way if the school is to rise above mediocrity;

ii) Use of "new mathematics curriculum" materials of any sort raises questions of cooperation among teachers, especially regarding articulation from one year to the next in the student's life. The question of faculty cooperation, also, must be resolved if a school is to achieve quality education;

iii) Although no cases have been observed, it seems safe to predict that if a school "adopts" any new curriculum materials without adequate teacher education, without adequate administrative support, or without an adequate determination to cooperate and to succeed, the results may well be abortive, disillusioning, and detrimental to further curriculum efforts of any sort;

iv) There is always a danger of "nullification by partial assimilation," "tokenism," or complacency. A school may pick up a small piece of new curriculum, and then, with a sigh of utter accomplishment, cease any further efforts. Curriculum evolution must, somehow, become an on-going activity for every teacher and every school.

In thinking about modern mathematics curricula, or in any curriculum modernization effort, it is worth reflecting on the remarks made by the NEA Project on Instruction[1] in relation to foreign language in the elementary school (FLES):

> Curriculum planners must weigh these arguments and the evidence on which they are based in deciding what priority to assign to foreign language study in the elementary school program. Language specialists have warned against faddish aspects of the FLES movement and have expressed dismay that in many communities FLES has been introduced without adequate preparation. A FLES program whether for selected groups or for all pupils, should be introduced only when the following conditions are met: "1) It is an integral and serious part of the school day; 2) it is an integral and serious part of the total foreign language program in the school system; 3) there is close articulation with later foreign language learning; 4) there are available FL specialists or elementary-school teachers with an adequate command of the foreign language; 5) there is a planned syllabus and a sequence of appropriate teaching materials; 6) the program has the support of the administration; 7) the high-school teachers of the foreign language in the local school system recognize the same long-range objectives and practice some of the same teaching techniques as the FLES teachers.
>
> In determining what priority is to be given to foreign language study in the elementary school, evidence about the particular

---

[1] Fraser, Sand, et al. (D1963).

> student body and the community the school serves should be studied. If the decision is to include foreign language in the elementary curriculum, the school that acts responsibly will meet the conditions stated above before instituting a FLES program.

One further remark is in order. The *Madison Project* has recommended a "light touch" in teaching Project materials, a consistent respect for each individual child, and a consistent effort to avoid embarrassing the child. In the elementary grades, at least up to grade 5, the Project recommends little or no overt "testing" of student behavior (cf. Sections 6 and 7 below), and the avoidance of expressing negative judgements to the child. If this practice is followed, there should be extremely little damage to students from any sense of failure, inadequacy, or embarrassment.

Psychiatrists and psychologists observing Project classes have repeatedly remarked upon the extremely high degree of student involvement, motivation, and participation -- "every child sitting on the edge of his chair" -- and at the same time the apparently extremely low anxiety level -- "no fear of failing," etc.

This approach should avoid many of the damaging effects that have often been associated with instruction in mathematics.

6. *Restraint in Evaluating Student Efforts.*

Hughes *Mearns* (cf. *Mearns*, C1958) has argued eloquently against teachers expressing judgements on poems that a child has written, worked hard on, laid bare his soul in, and (perhaps) come to feel proud of. Similar remarks are well known in art (Lindstrom, C1962). Analogous remarks are equally appropriate in mathematics, if we mean to go beyond a superficial rote memorization of addition facts, and the like. Strategies for attacking problems come in very great variety, and the teacher who would encourage student creativity must be prepared to try to understand some exceedingly novel strategies children devise (Rogers, C1961; Torrance, C1963).

The case of algorithms for subtraction could be cited as one example. The problem $64 - 28 = ?$ was attacked (and solved admirably) by a 3rd grade boy as follows:

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline {}^{-}4 \\ 40 \\ \hline 36 \end{array}$$

(cf. Davis, A1963h); this method was modified by some fifth graders as follows:

$$\begin{array}{r} 64 \\ -\underline{28} \\ 44 \end{array} \quad \text{(where 44 means } (4 \times 10) + ({}^{-}4 \times 1)\text{)}$$

(cf. the film entitled Education Report: The New Math); using a number line, some 6th graders made up this method:

"64 − 28" means "how far along the number line must I go to get to 64, if I start at 28?"

28 + 2 gets me to 30

+ 30 gets me to 60

+ 4 gets me to 64

$$\begin{array}{r} 2 \\ 30 \\ \underline{4} \\ 36 \end{array}$$

The teacher who wishes to encourage this sort of creativity -- which, to most mathematicians, is getting close to the essence of mathematics -- must listen carefully to the child, try to hear the sense in what the child says, respond with warm (and honest) appreciation to clever suggestions, and carefully avoid negative judgements on suggestions which appear to be in error. This, as has been discussed in various Project publications, does not mean ignoring reality; it means letting the child test out his suggestions by actual trial against mathematical reality.

In such an approach the teacher, who is not omniscient, must not attempt to seem so, nor must she appear to sit in judgement upon the child and his contributions.

For this reason, and several others, the Project attempts to avoid widespread testing of student achievement by the usual formal means.

A second reason for avoiding widespread testing of student achievement is that, originally conceived as a sampling of student behavior, such tests quickly become a beacon that many teachers steer by. (Hoffman, C1964.)

This becomes detrimental to the exploratory study of the world of mathematical ideas; it often leads to situations where "discovery," and even "ideas," are banished, and the emphasis is entirely on following rote directions that will produce a behavior which may seem to be the result of mathematical understanding, but in fact is not.

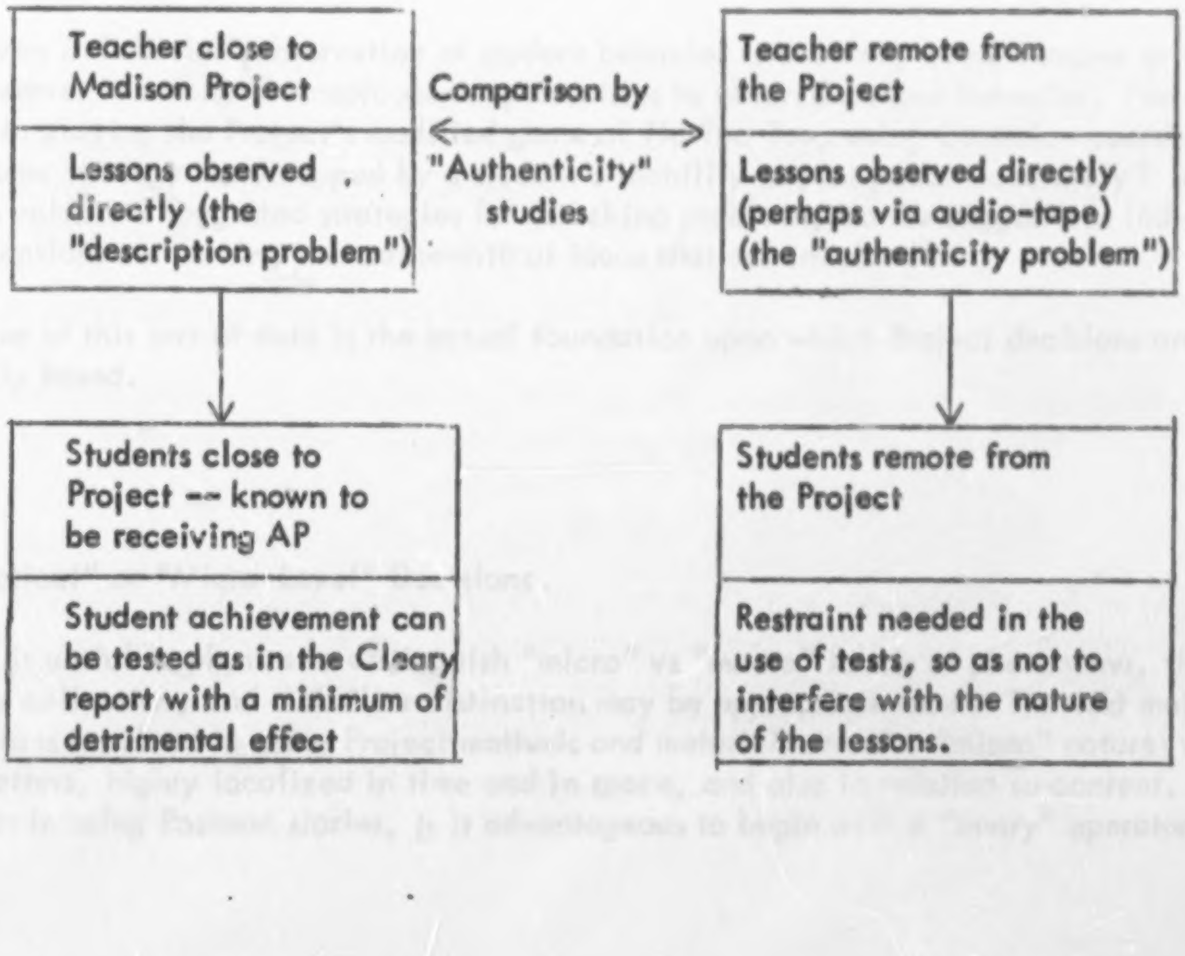
Mathematicians usually believe they can distinguish between the two if they can talk with the student (hence the prevalence of oral examinations in many mathematics departments), or if they can observe the student at work on new and unfamiliar problems. In these cases, one does in fact achieve something that approximates an actual sampling of student behavior, without giving the teacher and the student the beacon light that might guide them through rote channels.

Since our concern is with flexibility, originality, creativity , exploration, intuition, and divergent thinking, it is important to avoid pressures that would abolish all of these, in favor of a simple and "efficient" use of convergent thinking.

How, then, can we observe student behavior? The Project's answer is three-fold:

i) Tests, such as those used in the Cleary report (Cleary, A1965) can be administered to children studying the "authentic program" ("AP") with experienced teachers who work closely with the Project. These teachers will not change the nature of their lessons in order to give children extra rote preparation for the tests. Moreover, the actual classroom lessons of these children are under continual observation, so that what occurs in class is well known to many observers. Hence the test behavior of these children can be observed directly.

ii) For children in more remote classes, one can monitor the experiences they receive via the "authenticity problem." That is, diagrammatically, one has:


Teacher close to Madison Project
Lessons observed directly (the "description problem")
Comparison by
"Authenticity" studies
Teacher remote from the Project
Lessons observed directly (perhaps via audio-tape) (the "authenticity problem")
Students close to Project -- known to be receiving AP
Student achievement can be tested as in the Cleary report with a minimum of detrimental effect
Students remote from the Project
Restraint needed in the use of tests, so as not to interfere with the nature of the lessons.

Consequently, the "missing link" in the diagram can be inferred from studies such as the Cleary report, in children receiving the AP, and from "authenticity" studies, such as the tape panel.

Of course, the inference that the "remote" students are really receiving nearly the AP, and that the changes in their behavior will therefore approximate those of the children in the Project's own AP classes, depends upon the adequacy of the description of the AP program. If crucial variables have been overlooked in the description of the AP, then the experiences of the "remote" children may be quite different from those of the children in the Project's "AP" classes, and hence the effects upon the behavior of the remote students may be quite different.

iii) Consequently, some direct sampling of the behavior of remote students is desirable. This has not been attempted in the past, because of the concerns about restraint mentioned above, but is planned for the academic year 1965-6.

Still another approach to observing student behavior has been used, and will be described in the following section.

7. Inconspicuous Observation of Student Behavior.

Even if "official" observation of student behavior is avoided, every teacher or classroom observer has many inconspicuous opportunities to observe student behavior. For example, in playing the Project's modified game of Tic Tac Toe, using Cartesian coordinates, is the game strategy handicapped by a student's inability to plot points accurately? When students volunteer suggested strategies for attacking problems, do the suggestions indicate misconceptions concerning the mathematical ideas that are involved?

Use of this sort of data is the actual foundation upon which Project decisions are ordinarily based.

8. "Tactical" or "Micro-Level" Decisions.

It is useful in physics to distinguish "micro" vs "macro" levels of phenomena, theory, and data collection, and a similiar distinction may be appropriate here. The vast majority of decisions involved in using Project methods and materials are of a "micro" nature: very small matters, highly localized in time and in space, and also in relation to content. For example: in using Postman stories, is it advantageous to begin with a "unary" operator

$+{}^{+}5$, $-{}^{+}5$, $+{}^{-}3$, $-{}^{-}3$? If so, is it better to do so with explicit recognition to the children that this is a unary operator? Should the problem

$${}^{+}3 + {}^{+}5$$

be regarded as using a unary operator

$$+{}^{+}3$$

followed by a binary operator

$$(+{}^{+}3) + {}^{+}5\ ?$$

Is it better to begin with sums

$${}^{+}3 + {}^{+}5$$

or with products

$${}^{+}2 \times {}^{+}3\ ?$$

Is it better to introduce several kinds of problems simultaneously, thereby gaining the advantage of context, comparison, negative instances, and discrimination, or is it better to work with only one kind of problem at a time, thereby gaining simplicity (with a corresponding loss of over-all perspective and with a greater risk of creating mental sets and subsequent cognitive dissonance and confusion)?

There are several schools of thought among educational professionals concerning the best way to arrive at answers. Some believe that the systematic accumulation of minute studies will someday make teaching a science; others hold that the variables and alternatives are so numerous that a "systematic" attack is a snare and a delusion, and that teaching is best regarded as an art.

It is worth looking carefully at these minute decisions.

In the first place, the variables are in most cases interdependent. Thus, in an example above, informal evidence suggests that it may be advisable to introduce

$$+{}^{+}3$$

provided it is done unobtrusively and not mentioned explicitly, provided it is done before the students get to the classification of operations as unary, binary, etc., provided the teacher himself understands what he is doing and can avoid confusion, and provided the teacher feels comfortable with the amount of ambiguity that is involved. Hence, here, there are at least six variables:

use $+ {}^{+}3$, or avoid it

do it explicitly and clearly, or casually and unobtrusively

continue this notation into $\left[+ ({}^{+}3) \right] + {}^{+}5$, or not

do it before the student meets the explicit classification of operators as unary, binary, etc., or not

does the teacher feel comfortable with the amount of ambiguity that is involved?

In fact, there are surely many more, for there are questions of grade level placement, individual differences among students, type of classroom social organization used, classroom atmosphere, whether the problems are to be addition or multiplication problems first (or even, perhaps, both simultaneously), etc. Thus there are at least

$$2^5 \times 8 \times 5 \times 3 \times 2^2$$

possible alternatives -- that is to say, over 15,000 possibilities, and these relate to one very small question. It is entirely possible that, within a year or so, someone will create a better way to introduce children to the arithmetic of signed numbers, and "postman stories" will be dropped from the curriculum.

A decision on this level surely occurs at least once in every half-hour lesson, and in most cases, as here, the entire matter may disappear from the curriculum within a year or two.

The situation is similar to the proposal made -- with malice aforethought! -- by George Gamow. Suppose we imagine every page of a book in a reasonably uniform format: 30 lines (say), with 50 characters per line. Any book presently existing could be written using 24 lower case letters, 24 capital letters, the digits 0 1 2 3 4 5 6 7 8 9, a dozen marks of punctuation, perhaps 100 special symbols, and a space (even though this might require translating some ideas and symbols). There are then $30 \times 50 = 1{,}500$ characters per page (including spaces), and 159 possible entries for each of these 1,500 locations.

There are then

$$159^{1{,}500}$$

possible pages. For books of 500 pages or less, there are

$$\left(159^{1{,}500}\right)^{500}$$

possible books. Suppose we discontinue all of our laboratories, cease writing plays and novels, and merely print all

$$\left(159^{1{,}500}\right)^{500}$$

possible books.

These will include everything which could possibly be written in what we presently recognize as a book: every possible play, every possible novel, the entire history of the human race during its entire tenure on earth, a complete description of every possible cure for every possible disease, instructions for synthesizing every possible chemical, the design of every possible piece of machinery, and every possible piece of advise for every possible human dilemma that can be described in words.

Of course, a computer the size of a single small crystal, signaling at the speed of light, could not process this much information in 2^n times the life of the solar system, for some large value of n. The scheme is wholly impossible, because the numbers are far too large.

A similar situation exists with regard to solving the tower puzzle with 100 discs, although this seems, a priori, to be simple enough.[1]

Those who believe that "micro" decisions in education can be settled on a systematic basis are almost certainly committing the same error: the numbers are far too large; one cannot possibly deal with them by a straightforward systematic approach. We do need our poets, our playwrites, our ingenious inventors, and our teachers who can make creative use of their experience and their intuition. We cannot hope to proceed systematically in working with "micro" level decisions.

Fortunately, even an art can be studied, criticized, analyzed, and learned. It cannot, however, be replaced by a "systematic" approach. There isn't time.

---

[1] In fact, this would require

$$2^{100} - 1 \text{ moves.}$$

Using the approximation $2^{10} \doteq 1,000 = 10^3$, this is

$$10^{30} = 1,000,000,000,000,000,000,000,000,000,000$$

moves. Even at the rate of one move every millionth of a second, one would make less than 10^{14} moves per year. Hence, solving this puzzle must take more than

$$10^{30} \div 10^{14} = 10^{16} \text{ years,}$$

that is, longer than

$$10,000,000,000,000,000 \text{ years.}$$

There is an ancient legend that some monks are working to solve this puzzle in a temple in the city of Hanoi, and that when they finish the world will come to an end. The real question is, can the solar system possibly last long enough to give the monks enough time to finish solving the puzzle?

XI. CONCLUSIONS AND IMPLICATIONS

In discussing conclusions and implications, it is important to emphasize that a "demonstration project" such as the present one is neither an inductive laboratory study nor a deductive argument via formal logic. It is a practical effort in actual school situations. Consequently, its "conclusions" are summaries and interpretations of accumulated practical experience. This does not mean thay are worthless, nor even worth less, but they must be seen realistically for what they are.[1]

A. Remarks relating to the General Idea of Progress

i) Devising or interpreting a school mathematics program involves assumption about the nature of knowledge, about the nature of schools, about the nature of children, and about the nature of our society. Some of the disagreements and differing emphases concerning school mathematics programs are direct consequences of different assumptions in these four crucial areas. There may be considerable gain in attempts at making these assumptions more explicit and examining them more critically.

ii) Looking at school mathematics programs in 1965, it is clear that some genuine progress has been made, but that in general this progress has not been very great.[2] There has not been any "revolution." Even the pace of "revolution" is exceedingly slow, and for many schools and many communities the rate of change is essentially zero.[3]

---

[1] Cf. Conant (D1965)

[2] Cf. Fraser, Sand, et al. (D1963), pp. 197 ff.

[3] It is interesting to compare Mayer (B1963) with Gross and Murphy (B1964). Mayer visited and describes many of the same schools and projects that Gross and Murphy did, yet Mayer's tone is pessimistic if not alarm-sounding, whereas Gross and Murphy are optimistic nearly to the point of euphoria. Even allowing for the expected differences in personal responses of different writers, the contrast is striking -- and, as it happens, significant, since Gross and Murphy were in effect asking, "Are we finding better ways to help children learn?" whereas Mayer was, in effect, asking, "Are recent improvements being utilized in actual school situations on a broad scale?"

iii) The Madison Project's methods and materials can be divided into two groups: "stable" materials that have been tried by many teachers and learned by many students with reasonable reliability, and "exploratory" materials that have not reached stable forms (and may never do so). In some cases "exploratory" materials can be made "stable" once their format is reasonably perfected and adequate teacher aids are developed; in other cases the "exploratory" topics work satisfactorily only with a few children, with a few teachers, and in a few schools.

B. Remarks Relating Primarily to Procedure.

i) A "bits-and-pieces" approach, where rather small units are developed for insertion into the school program, is possible and does make some contribution. Indeed, if curriculum evolution is to become an on-going activity of any significant vitality, it is hard to imagine any alternative to some use of a "bits-and-pieces" approach.

Having said this, it should be added that educational practice in recent years has tended to develop definite "slots" in the curriculum, such as "10th grade geometry" or "9th grade algebra," or" a methods course for secondary mathematics teachers," and the common practice is to seek textbooks which are designed to fit rather neatly into these slots, with a minimum of overlap or unoccupied space.[1] This tendency markedly inhibits any serious curriculum work at the school level. (We shall return to this matter again below.)

ii) The Madison Project does not use the common procedure of teaching mathematical ideas to teachers, and then leaving the teachers with the nearly impossible task of translating these new and unfamiliar ideas into suitable learning experiences for children. Instead, the Project uses a procedure whereby Ph.D mathematicians (or their approximate equivalent) work in classrooms with actual children, and devise suitable experiences from which the children can learn the desired concepts. These fully-worked-out actual classroom lessons are recorded or transmitted in various ways, often via films showing actual classroom lessons, or via demonstration classes.

This procedure has a great deal of power. It does not, incidentally, detract from the role of the teacher; it merely gives the teacher more food for thought, and an enlarged set of available educational tools. It is tantamount to an attempt to advance the state of the art.

---

[1] Cf. Fraser, Sand, et al. (D 1963), p. 201 and elsewhere.

iii) As suggested above, the role of films showing actual classroom lessons can become one of the most powerful tools available in curriculum work, and undoubtedly in many other aspects of the study of educational processes.

iv) It is possible to identify various basic organizing concepts of mathematics that are conspicucusly missing from the traditional school program at appropriate grade levels. By the procedures indicated above, some of these basic organizing concepts can be added to the school program. (The use of Cartesian graphs, the study of functions, and the use of variables are three cases in point.)[1]

C. Difficulties

i) It is unhappily evident that elementary school teachers are often not at home either using or discussing the basic concepts of mathematics, such as the correct use of variables, or the plotting of points on Cartesian coordinates, etc. On the other hand, it is equally clear that programs to remedy this lack can be created and can operate successfully (cf. Section E, below).

ii) Elementary school rationale has often given great priority to the proposition that every question has an answer, which is both unique and precise: one and only one answer, without error or ambiguity. Such a view is incompatible with science, with mathematics—or, for that matter, with nearly any serious body of thought. In fact many questions have no answer, some questions have many (equally good) answers, and some questions have approximate answers but no perfect answers. The "tolerance of ambiguity" that is required of anyone who would see the world realistically is a severe demand for some teachers.

iii) Closely related to this is the fact, eloquently discussed by Schwab, Torrance, and others,[2] that elementary schools often operate both in practice and in theory, within the framework of a "rhetoric of conclusions" which is absoulte, unchanging, and inevitably subject to frequent error, instead of what would seemingly be more appropriate (and is more common at more advanced educational levels), a "rhetoric of inquiry." The appreciation of tentativeness, open-endedness, and inquiry that is recommended in items ii and iii can probably be achieved in one way only--by making this dimension far more prominent in the education of teachers themselves.

---

[1] For evidence supporting this view, cf. for example the films: Guessing Functions, A Lesson with Second Graders, Graphing a Parabola, Graphing an Ellipse, etc.

[2] Cf. Schwab (C 1964), and Fraser and Sand, et al. (D 1963), p.41, and elsewhere.

iv) There are many unsettled questions about the social organization of the classroom. Should children work alone, or in pairs, or in small "committees," or should the entire class work together? What balance is desirable in a mathematics program between individual work, small group work, and participation in a total class discussion? What balance is desirable in a mathematics program between reading, writing, seminar-type discussion, lectures, individualized instruction, use of programmed materials, individualized work on projects, group work on projects, or total class cooperation on a project? What forms of noise level and student activity are appropriate?

Studying these alternatives poses a question of "which comes first, the chicken or the egg": without well-developed materials designed for the approach in question, that approach cannot be tested in practice. However, with little information on various approaches, it is difficult to select suitable objectives for a program of that type, and to devise appropriate materials for it.

School programs appear to have followed closely a reading-writing-reciting-lecture approach, and to have given relatively little serious thought to alternative types of activities for a mathematics program. At the very least, this is probably detrimental to some students and tends to eliminate some kinds of mathematical topics.

As an example of one question: is there any relation between the freedom and responsibility that characterize general student social behavior, and the kinds of creativity, originality, initiative and judgement that the students exhibit in their mathematical work?

v) Articulation between elementary school and secondary school is, in most actual situations, a good deal less than ideal. Relatively few secondary teachers have any real interest in elementary school children or elementary school programs, and do not approach the elementary school sympathetically and with understanding. In fact, some of the most creative teaching to be seen in today's schools occurs at the elementary level, but more work is needed to get secondary teachers aware of these, and eager to articulate their programs with the elementary programs.[1]

---

[1] The remarks concerning the FLES program that are made in Fraser, Sand et al.(D 1963), pp.144-145, are eqaally relevant concerning mathematics.

vi) Data collection and study concerning the educational aspects of a school mathematics program depend upon many assumptions about appropriate methodology, appropriate kinds of questions, appropriate kinds of evidence, and appropriate kinds of answers. There are many different viewpoints, and choosing suitable strategies for learning about school mathematics programs is largely an unsettled question at present.[1]

vii) Concerning the more sophisticated "non-stable" portions of the Madison Project's methods and materials, it has been observed that these work extremely well in some schools, so that students (for example) may begin their study of these materials in grade 3, and continue eagerly until grades 8 or 9, by which time they have learned a very large amount of mathematics and have developed a very powerful attack on mathematical problems. In other schools these "sophisticated" parts of the Project's materials do not work well at all. The difference is not understood at present. (It is however undoubtedly true that on today's educational landscape the highest levels of quality are to be seen only now and then.)

D. Remarks Concerning Unsettled questions.

i) It has been noted that the "sophisticated" portions of Madison Project materials work well in some schools, but poorly in others. Considerable light would undoubtedly be shed on the nature of "quality" in educational practice, and the conditions which accompany it, if the differences in these two types of schools could be described more adequately.

ii) Several additional "unsettled questions" can be inferred from the preceding remarks.

E. Some Conclusions.

i) The successful operation for the past 4½ years of the Webster College undergraduate program to educate elementary school mathematics specialists by a program emphasizing a "rhetoric of inquiry," and including 30 credits of college-level mathematics, demonstrates unequivocally that such a program is possible, that it can turn out a very strong graduate, that it can recruit suitable students, and that its graduates can find appropriate employment. It also demonstrates dramatically that the range of variety among the nation's teacher-education programs is unnecessarily small, and omits the serious exploration of many promising approaches.

---

[1] Cf. George G. Stern, "Measuring Noncognitive Variables in Research on Teaching," in Gage (D 1963) pp. 398-447; Conant (D 1965); Hunt (D1961); Stanton and Schwartz (D 1954); Redl and Wineman (D 1957); Buxton (D 1962); and other references in the bibliography.

ii) The successful operation for the past 2½ years of the Webster College MAT program to educate experienced elementary teachers who wish to become mathematics specialists, via a program similar to the one described in item i, demonstrates that such a program can operate successfully and can be of great value. Again, the lack of variety in generally-available programs is dramatically emphasized.

iii) Although the program has been built thus far by Burt Kaufman, with no Madison Project involvement until recent weeks, it is worth citing that the program at Nova High School indicates unmistakably that many high school students can study almost the entire range of high school mathematics content on their own, outside of school, and devote their high school work to the study of mathematics ordinarily thought to be of college level; the students handle this work with zest and originality, and appear to prefer this to the overelaboration of the trivial and the obvious which has ordinarily consituted the high school mathematics program. Again, it becomes clear that the range of possible school programs has hardly been explored.

iv) Curriculum planning and building in schools hardly exists, at present, as a serious activity. "Curriculum planning" in schools today is usually limited to glancing at various textbooks and "making an adoption." The school mathematics program could be greatly strengthened if serious curriculum planning at the school level received a significant share of a school's effort, time, and money.[1]

v) The individual teacher needs a larger role in curriculum planning, and opportunities to achieve the level of understanding that is needed to play this role effectively. Indeed, we would argue that if a teacher does not have the competence to help share the curriculum in significant ways, then he probably does not have the competence to teach the curriculum when it is selected.[2]

---

[1] Cf. Fraser, Sand, et al. (D 1963), p. 205, and elsewhere.

[2] Again, cf. Fraser, Sand et al. (D 1963), p. 205 and elsewhere.

vi) If a school is planning to proceed with new mathematics, it needs to make a determined effort, backed by a significant commitment. What is involved is much deeper than merely "a new textbook adoption." What is required is more effort at local curriculum planning, a significant teacher-education program, etc. Two of the biggest dangers facing mathematics curriculum work at present are nullification by partial assimilation (i.e., "tokenism" and complacency), and the hasty adoption of goals without a determination to provide what is necessary to move effectively toward those goals.

vii) Organizations aimed at producing effective operational contributions to curriculum evolution are needed, and will continue to be needed. One of the best at present is Educational Services Incorporated, in Watertown, Massachusetts. Neither ordinary university and college operations, nor ordinary school operations, meet this need in any significant way.

viii) Anyone interested in knowing exactly what kinds of mathematics experiences are included in "Madison Project methods and materials" can best find this out by viewing some of the films showing actual classroom lessons.

XII. BIBLIOGRAPHY

A. Publications by the Madison Project, or relevant publications by individuals associated with the Project:

Cleary, J. Robert. (1965), "A Study of Test Performance in Two Madison Project Schools and One Control School," available from the Madison Project.

Clem, Gordon H. (1963), "The Madison Project." Report of the Annual Conference of the National Association of Independent Schools (June, 1963), pp. 46 - 49.

Cochran, Beryl S. (1962), "The New Math," Connecticut College Alumnae News (December, 1962), pp. 11 - 13.

Cohen, Donald. (1964), "A Lesson on Absolute Value," The Arithmetic Teacher, Vol. 11, No. 8 (December, 1964), pp. 561 - 562.

----------. Independent Exploration Materials/Grades 5 - 9, No. 1, Peg Game, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 2, Weights and Spring, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 3, Tower Puzzle, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 4, Pulley, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 5, Pendulum -1, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 6, Conic Sections - 1, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 7, Geoboard, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 8, Conic Sections - 2, the Madison Project (in preparation).

----------. Independent Exploration Materials/Grades 5 - 9, No. 9, Pendulum - 2, the Madison Project (in preparation).

Cohen, Donald. Independent Exploration Materials/Grades 5 - 9, No. 10, Geometry - 1, the Madison Project (in preparation).

Davis, Robert B. (1955), "Emotion and Thought," The Mathematics Teacher, Vol. XLVIII, No. 3 (March, 1955), pp. 133 - 142 (reprinted in M. Vere Devault's Improving Mathematics Programs, Merrill Books, Columbus, Ohio, 1961, pp. 108 - 121).

----------. (1958), "The Mathematics Revolution: Causes and Directions," Frontiers of Secondary Education, Syracuse University Press, Chapter 7.

----------. (1959a), "How Do We Learn Mathematics?" MIT Technology Review, Vol. LXII, pp. 28 - 30.

----------. (1959b), "Seventh Grade Algebra: The Madison Project," New York State Mathematics Teachers Journal, Vol. IX, No. 2, pp. 5 - 8.

----------. (1959c), Solving Problems and Constructing Systems -- Quadratic Equations and Vectors. Report of an Orientation Conference for SMSG Experimental Centers, Chicago, Illinois (September 19, 1959), pp. 97 - 101.

----------. (1959d), "Special Report -- The Road Between," Updating Mathematics (October, 1959), Section 1.

----------. (1960a), "The Syracuse University 'Madison Project'," American Mathematical Monthly, Vol. LXVII, No. 2 (February, 1960), pp. 178 - 180.

----------. (1960b), "Mathematics For Younger Children -- The Present Status of the Madison Project," New York State Mathematics Teachers Journal, Vol. X, No. 2 (April, 1960), pp. 75 - 79.

----------. (1960c), "The 'New' Pre-College Mathematics," Undergraduate Physics and Mathematics in Electrical Engineering, Proceedings of the Sagamore Conference, September, 1960, Department of Electrical Engineering, Syracuse University.

----------. (1960d), "Algebra in Grades Three Through Ten: The Madison Project," Grade Teacher, Vol. LXXVIII (October, 1960), pp. 34 +.

----------. (1961), "The Madison Project: Algebra in Grades 3 - 9," Frontiers of Elementary Education, Syracuse University Press, pp. 56 - 59.

----------. (1962a), A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project. Report submitted to the Commissioner of Education, U.S. Department of Health, Education and Welfare.

Davis, Robert B. (Continued)

----------. (1962b), Axioms for Arithmetic and Algebra. The Madison Project.

----------. (1962c), The Madison Project: A Brief Introduction to Materials and Activities. The Madison Project (revised 1965).

----------. (1962d), Notes on the Film: First Lesson (booklet to accompany film of the same name). The Madison Project.

----------. (1962e), Notes on the Film: A Lesson with Second Graders (booklet to accompany film of the same name). The Madison Project.

----------. (1962f), Notes on the Film: Matrices (booklet to accompany film of the same name). The Madison Project.

----------. (1962g), "Algebra in Grades Four, Five and Six," Grade Teacher, Vol. LXXIX (April, 1962), pp. 57 +.

----------. (1962h), "Report of the Madison Project," Science Education News (December, 1962), pp. 15 - 16 (a publication of the American Association for the Advancement of Science).

----------. (1963a), A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. The Madison Project. Report submitted to the Commissioner of Education, U.S. Department of Health, Education and Welfare.

----------. (1963b), "Mathematical Thought and The Nature of Learning: The Madison Project View," Frontiers of Education. Report of the Twenty-Seventh Educational Conference sponsored by the Educational Records Bureau. American Council on Education, Educational Records Bureau, 21 Audubon Ave., New York 32, New York, pp. 79 - 83.

----------. (1963c), Matrices, Functions, and other Topics: A Student Discussion Guide. The Madison Project.

----------. (1963d), Matrices, Functions, and other Topics: A Text for Teachers. The Madison Project.

----------. (1963e), "The Evolution of School Mathematics," Journal of Research in Science Teaching. Vol. 1, pp. 260 - 264.

----------. (1963f), "The Syracuse University - Webster College Madison Project," in An Analysis of New Mathematics Programs, edited by Philip Peak, et al. NCTM, pp. 16 - 19; 20 - 21.

Davis, Robert B. (Continued)

----------. (1963g), "Three Periods of Education in the United States," Christian Science Monitor (January 12, 1963), p. 11.

----------. (1963h), "Math Takes a New Path," The PTA Magazine, Vol. LVII, No. 6 (February, 1963), pp. 8 - 11.

----------. (1963i), "How Is the Madison Project Different?" Illinois Council of Teachers of Mathematics News Letter, Vol. XIV, No. 2 (May, 1963), pp. 7 - 8.

----------. (1963j), "Report on Madison Project Activities, September, 1962 - November, 1963," a report submitted to the National Science Foundation (December, 1963).

----------. (1964a), Discovery in Mathematics: Student Discussion Guide. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.

----------. (1964b), Discovery in Mathematics: A Text for Teachers. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.

----------. (1964c), "The Madison Project's Approach to a Theory of Instruction," Journal of Research in Science Teaching, Vol. 2, pp. 146 - 162.

----------. (1964d), Undergraduate College Education in Relation to the 'New Curriculum' Projects," Current Issues in Higher Education, pp. 153 - 155.

----------. (1964e), "Report on the Syracuse University - Webster College Madison Project," American Mathematical Monthly, Vol. 71, No. 3 (March, 1964), pp. 306 - 308.

----------. (1964f), "Some Remarks on a Theory of Instruction," Piaget Re-discovered. Edited by Verne N. Rockcastle and Richard E. Ripple. A report of the Conference on Cognitive Studies and Curriculum Development (March, 1964). School of Education, Cornell University, pp. 134 - 138.

----------. (1964g), Experimental Course Report/Grade Nine (Report No. 1, June, 1964), available from the Madison Project.

----------. (1964h), "Guidelines for the Assessment, Initiation, and Reassessment of Curriculum Programs" (paper presented at the Chicago ASCD meeting, Fall, 1964).

----------. (1964i), "Goals for School Mathematics: The Madison Project View," Journal of Research in Science Teaching, Vol. 2, Issue 4 (December, 1964), pp. 309 - 315.

Davis, Robert B. (Continued)

----------. (1965a), "Mathematics," Chapter 6 in: Glenys Unruh, editor, New Curriculum Developments, ASCD, pp. 46 - 56.

----------. (1965b), "Some Remarks on 'Learning By Discovery'," available from the Madison Project.

----------. (1965c), "Recent Activities of the Madison Project," American Mathematical Monthly (January, 1965).

----------. (1965d), "New Mathematics is Deeper Than Mathematics," Five Views of the "New Math," Council for Basic Education, Washington, D.C. (April, 1965), pp. 10 - 12.

----------. Matrices, Logic and other Topics: Student Discussion Guide. Addison-Wesley Publishing Company, Reading, Massachusetss (in preparation).

----------. Matrices, Logic and other Topics: A Text for Teachers. Addison-Wesley Publishing Company, Reading, Massachusetts (in preparation).

Gold, Sheldon. (1965), The Reaction of Teachers to the Methods of Teaching Modern Mathematics Using Madsion Project Materials. School of Education, the City College of the City University of New York (January, 1965).

Hopkins, Charles D. (1965), "An Experiment on Use of Arithmetic Time in the Fifth Grade" (unpublished doctoral thesis). School of Education, Indiana University (June, 1965).

In-Service Course I. (1964), the Madison Project.

In-Service Course II. (1965), the Madison Project.

Kharas, Katherine, et al. (1964), Mathematics. Prepared at the Entebbe Mathematics Workshops. Secondary Two, Vol. 3 (March, 1964). African Education Program, Educational Services Inc., Watertown, Massachusetts.

Machtinger, Doris Diamont. Experimental Course Report/Kindergarten. (Report No. 2, July, 1965). The Madison Project (in preparation).

----------. Experimental Course Report/First Grade. The Madison Project (in preparation).

Newsletter No. 1 (July, 1965). The Madison Project.

B. The following publications refer to the Madison Project:

Allen, Frank B., et al. (1961), The Revolution in School Mathematics, National Council of Teachers of Mathematics.

Bergamini, David and the editors of Life Magazine. (1963), Mathematics. Life Science Library Series. Time Inc., New York. pp. 193 - 196.

Chamberlain, John. (1960a), "Problems in Education," Where Do We Go From Here? Available from Dow Jones and Co., Inc.

----------. (1960b), "The Third 'R' -- How Some Schools Speed Up Teaching of Arithmetic," Wall Street Journal (February 9, 1960), Editorial page.

----------. (1960c), "Philosophies of Education. Pragmatic Parents Force a Turn to Multiple Track System," Wall Street Journal (February 23, 1960), Editorial page.

Clarkson, David. (1962), "Taxicab Geometry, Rabbits, and Pascal's Triangle-Discoveries in a Sixth-Grade Classroom," The Arithmetic Teacher, Vol. IX, No. 6 (October, 1962), pp. 308 - 313.

Crawford, Douglas H. (1962), "Recent Developments in School Mathematics -- Further Trends in the U.S.A." Mathematics Teaching, No. 19 (Summer, 1962), pp. 5 - 13. (The bulletin of the Association for Teaching Aids in Mathematics; c/o Ian Harris, 122 North Road, Dartford, Kent, England.)

Dienes, Zoltan P. Current Work on the Problems of Mathematics Learning. International Study Group for Mathematics Learning, 200 California Ave., Palo Alto, California.

----------. The United Nations Educational, Scientific, and Cultural Organization Curriculum Study (UNESCO), University of Adelaide, Adelaide, Australia.

Dorothea, Sister M. (1962), "Mathematics Refresher Course for Elementary School Teachers," New York State Mathematics Teachers Journal, Vol. XII, No. 2 (April, 1962), pp. 73 - 74.

ESI Quarterly Report (1965), Vol. III, No. 3 (Summer - Fall, 1965).

Footlick, Jerrold. (1965), "The New Math -- Philosophy of an Abstract Art," The National Observer (March 1, 1965), pp. 12 - 13.

Goals for School Mathematics. (1963), The Report of the Cambridge Conference on School Mathematics, Houghton Mifflin Co., Boston, Massachusetts.

Goodlad, John I. School Curriculum Reform in the United States. Revised edition (now in preparation).

Gross, Ronald. (1964), "Two-Year-Olds Are Very Smart," The New York Times (September 6, 1964), Magazine Section, pp. 10 - 11.

Henderson, Kenneth B. (1963), "Mathematics," Chapter 6 in Using Current Curriculum Developments, ASCD, pp. 49 - 58.

Jackson, S.B. (1962), "The School Mathematics Study Group Elementary School Program," New York State Mathematics Teachers Journal, Vol. XII, No. 2 (April, 1962), pp. 66 - 72.

Jonas, Arthur. (1962) , New Ways in Math. Prentice-Hall.

----------. (1964) , More New Ways in Math. Prentice-Hall.

Jones, Phillip S. and Arthur F. Coxford, Jr. (1964), "Academic and Professional Preparation of Secondary School Mathematics Teachers," Chapter VI in Review of Educational Research, Vol. XXXIV, No. 3 (June, 1964), Natural Sciences and Mathematics, pp. 322 - 333.

Lansdown, Brenda. (1963), "New Perspective in Teacher Education," New York State Mathematics Teachers Journal, Vol. XIII, No. 1 (January, 1963), pp. 8 - 16.

Lockard, J. David (editor). (1965), Third Report of the Information Clearinghouse on New Science and Mathematics Curricula (March, 1965), AAAS and the University of Maryland.

Lutz, Marie E. (1962), "Experimental Program for Seventh Grade," New York State Mathematics Teachers Journal, Vol. XII, No. 3 (June, 1962), pp. 112 - 114.

Maxey, David R. (1963), "Why Father Can't Do Johnny's Math," Look Magazine (November 5, 1963), pp. 85 - 91.

Mayer, Martin. (1963), "Last Chance for Our Schools?" Saturday Evening Post (September 14, 1963), pp. 24 ff.

Murphy, Judith and Ronald Gross. (1964), "There's No Limit to Learning When Children's Natural Curiosity is Encouraged," Parents' Magazine, Vol. XXXIX, No. 2 (February, 1964), pp. 62 - 112.

Murphy, Judith. (1965), Middle Schools. Educational Facilities Laboratories, 477 Madison Avenue, New York, New York 10022.

Parsons, Cynthia. (1960), "Algebra as Presented to Fourth Graders is Grasped with Enthusiasm," Christian Science Monitor (January 9, 1960), p. 7.

Peak, Philip, et al. (1963), An Analysis of New Mathematics Programs. NCTM.

Rockcastle, Verne N. (editor), et al. (1964), Piaget Re-discovered. A report of the Conference on Cognitive Studies and Curriculum Development (March, 1964), School of Education, Cornell University (edited by Verne N. Rockcastle and Richard E. Ripple).

Sharp, Evelyn. (1964), A Parent's Guide to the New Mathematics. E. P. Dutton and Company.

Social Science Research Council. (1965), Items. Vol. 19, No. 1 (March, 1965), (230 Park Avenue, New York, New York 10017). p. 10.

Studies in Mathematics Education. (1960), Scott, Foresman and Co. (Summer, 1960).

Struthers, Joseph, et al. Mathematics for the Elementary School. An In-Service Course for Elementary School Teachers. Colorado Department of Education, Office of Instructional Services, Division of Elementary and Secondary Edcuation, Denver, Colorado 80203.

Wyant, William K., Jr. (1961), "The 'New Math'," St. Louis Post-Dispatch (November 5, 1961), pp. 10 - 15.

Zacharias, Jerrold R., et al. (1964), Innovation and Experiment in Education (March, 1964). Report submitted to the President's Science Advisory Committee. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402 (price 35 cents).

C. The following publications do not refer specifically to the Madison Project, but do present a rationale, or the results of studies, that are generally similar to or supportive of the Project's rationale and methods.

Alexander, William M. (1964), Changing Curriculum Content. Association for Supervision and Curriculum Development, Washington 36, D.C.

Alexander, William M. and Lloyd S. Michael. (1965), "Current Curriculum Development: Problems and Prospects," Chapter 11 in New Curriculum Developments, edited by Glenys Unruh, ASCD, pp. 96 - 104.

Ashton-Warner, Sylvia. (1963), Teacher. Simon and Schuster, New York.

Avers, Paul W. (1964), "A Unit in High School Geometry Without the Textbook," The Mathematics Teacher, Vol. LVII, No. 3 (March, 1964), pp. 139 - 142.

Berne, Eric. (1964), Games People Play, Grove Press.

Blackwood, Paul E. (1963), "Science," Chapter 7 of Using Current Curriculum Development, ASCD, Washington 36, D.C., pp. 59 - 70.

Boole, Mary Everest. (1904), The Preparation of the Child for Science, Oxford.

Boulle, Pierre. (1957), The Test, Vanguard Press. Popular Library edition (paperback), 1960.

Brecher, Ruth and Edward Brecher. (1962), "Gifted Children Need Freedom to Learn," Parents' Magazine (June, 1962), pp. 44 ff.

Bruner, Jerome S. (1963a), On Knowing: Essays for the Left Hand. Harvard University Press, pp. 12 - 15; pp. 18 - 20.

----------. (1963b), The Process of Education. Harvard University Press, pp. 12 - 14 and p. 25.

----------. (1963c), "Needed: A Theory of Instruction," Educational Leadership, Vol. 20, No. 8 (May, 1963), pp. 523 - 532.

----------. (1963d), Growing (presentation at Invitational Conference on Testing Problems, November 2, 1963). Educational Testing Service, Princeton, New Jersey.

----------. (1964), "The Cause of Cognitive Growth," American Psychologist, Vol. 19, No. 1, pp. 1 - 15.

Cantor, Nathaniel. (1961), The Dynamics of Learning. Henry Stewart, Inc., New York.

Coddington, Earl A. (1963), "Scholastic Aptitude Tests in Mathematics," The American Mathematical Monthly (August - September, 1963), pp. 750 - 755.

Combs, Arthur W., Earl C. Kelley, Carl R. Rogers, A.H. Maslow, et al. (1962), Perceiving, Behaving, Becoming, ASCD.

Courtis, Stuart A. (1963), "Our Choice: Revolution or Destruction," Educational Leadership, Vol. 20, No. 8 (May, 1963), pp. 520 - 522.

Cronbach, Lee J. (1954), Educational Psychology (Chapter 17, "Reaction to Thwarting," and Chapter 18, "Personal Integration"). Harcourt, Brace and Co., New York.

Day, Robert C. and Robert L. Hamblin. (1964), "Some Effects of Close and Punitive Styles of Supervision," American Journal of Sociology, Vol. LXIX, No. 5 (March, 1964), pp. 499 - 510.

de Charms, Richard, Virginia Carpenter and Aharon Kuperman. "The 'Origin-Pawn' Variable in Person Perception," Sociometry (to appear).

Dexter, Lewis Anthony. (1964), The Tyranny of Schooling. Basic Books.

Dienes, Z.P. (1964), The Power of Mathematics. Hutchinson Educational Limited, London, especially pp. 17 - 25.

Evan, William M. and Morris Zelditch, Jr. (1961), "Experiment on Bureaucratic Authority," American Sociological Review, Vol. 26, No. 6 (December, 1961), pp. 883 - 893.

Faber, Norman, Phyllis Gross, and Robert Hansen. (1964), Mathematics and Living Things. School Mathematics Study Group (SMSG), Stanford University.

Flanders, Ned A. (1959), "Teacher-Pupil Contacts and Mental Hygiene," Journal of Social Issues, Vol. XV, No. 1, pp. 30 - 39.

Fletcher, T.J. (editor). (1964), Some Lessons in Mathematics, Cambridge University Press.

Gilchrist, Robert S. and Donald Snygg. (1965), "The Case for Change," Chapter 1 in New Curriculum Developments, edited by Glenys Unruh, ASCD, pp. 1 - 9.

Goodman, Paul. (1964), Compulsory Mis-education. Horizon Press, New York. pp. 53, 54, and p. 81.

Gross, Ronald. (1964), "Reforms and Innovations in Education to Stimulate Learning in Lower Grades" (results of recent findings on development of intelligence). New York Times (September 6, 1964), Sec. 6, pp. 1 - 10.

----------. (1965), "What We Don't Know May Help Us," New York Herald Tribune, Book Week (February 7, 1965), pp. 5 and 15.

Gunther, M. (1964), "Cracking the Grown-Ups' Code," Saturday Evening Post (June 20, 1964), pp. 34 - 35.

Harlow, H. F. (1932), "Social Facilitation of Feeding in the Albino Rat," Journal of Genetic Psychology, Vol. XLI, No. 1 (September, 1932), pp. 211 - 221.

Harlow, H.F., M.K. Harlow, and D.R. Meyer. (1950), "Learning Motivated by a Manipulation Drive," Journal of Experimental Psychology, Vol. 40, pp. 228 - 234.

Hawkins, David. (1964), "On Living in Trees," Karl Muenzinger Memorial Lecture, University of Colorado.

Henry, Jules. (1963), "American Schoolrooms: Learning the Nightmare," Columbia University Forum. Vol. 6, No. 2 (Spring, 1963), pp. 23 - 30.

Hoffman, Banesh. (1964), The Tyranny of Testing. Collier Books, New York.

Holt, John. (1964), How Children Fail. Pitman.

Hunt, J. McV. (1964), "Revisiting Montessori," in Maria Montessori, The Montessori Method, Schoken Books Inc., pp. xi - xxxix.

Huxley, Aldous. (1964), "The Education of an Amphibian," contained in Tomorrow and Tomorrow and Tomorrow, Signet Books.

Kaplan, Jerome D. (1964), "An Example of Student-Generated Sequences in Mathematics Instruction," The Mathematics Teacher, Vol. LVII, No. 5 (May, 1964), pp. 298 - 302.

Kelley, Earl C. (1951), The Workshop Way of Learning. Harper, New York.

Kersh, Bert Y. (1964), "Learning By Discovery: What Is Learned?" The Arithmetic Teacher, Vol. 11, No. 4 (April, 1964), pp. 226 - 232.

Lewin, Kurt. (1942), "Field Theory in Social Science," Chapter IV, Field Theory and Learning, pp. 60 - 86.

Lindstrom, Miriam. (1962), Children's Art. University of Chicago Press.

Macdonald, I.D. (1964a), "Matrices as 'New Mathematics' I," The Australian Mathematics Teacher, Vol. 20, No. 1 (April, 1964), pp. 12 - 16.

----------. (1964b), "Matrices as 'New Mathematics' II," The Australian Mathematics Teacher, Vol. 20, No. 2 (July, 1964), pp. 29 - 35.

Mearns, Hughes. (1958), Creative Power. Dover Publications, New York.

Neill, A.S. (1960), Summerhill. Hart Publishing Co., New York.

Page, David and UIÇSM Staff. (1960), "Arithmetic with Frames." Available from the University of Illinois Arithmetic Project, 1207 West Stoughton St., Urbana, Illinois.

Page, David. (1960), "Do Something About Estimation!" Updating Mathematics, Vol. 2, No. 8 (April, 1960).

----------. (1962a), Maneuvers on Lattices. University of Illinois Arithmetic Project.

----------. (1962b), "Well-Adjusted Trapezoids," Updating Mathematics, Croft Educational Services, 100 Garfield Ave., New London, Connecticut, Vol. 4, No. 8, Sec. 11.

----------. (1964), Number Lines, Functions, and Fundamental Topics. Macmillan.

Paugh, Mildred L. (1962), "A Journey into Mathematical History," New York State Mathematics Teachers Journal, Vol. XII, No. 3 (June, 1962), pp. 120 - 125.

Pincus, Morris. (1964), "An Adventure in Discovery," The Arithmetic Teacher, Vol. 11, No. 1 (January, 1964), pp. 28 - 29.

Polya, George. (1954), Mathematics and Plausible Reasoning. Two Vols., Princeton University Press.

----------. (1957), How to Solve It. Doubleday, New York.

----------. (1962), Mathematical Discovery. Two Vols. John Wiley and Sons, Inc. (Vol. 1, 1962; Vol. II, 1965).

----------. (1963), "On Learning, Teaching, and Learning Teaching." American Mathematical Monthly, Vol. 70, No. 6, pp. 605 - 619.

Primary Education: Changing Dimensions. (1965), edited by Evelyn Weber and Sylvia Sunderlin, published by Association for Childhood Educational International, 3615 Wisconsin Ave., N.W.,Washington,D.C. 20016. Especially the article by Ronald C. Welch, " 'New Mathematics' in the Primary Grades," pp. 44 - 55.

Reik, Theodor. (1948), Listening With the Third Ear. Farrar, New York.

Richmond, Donald, A. L. Putnam, et al. (1963), Basic Concepts of Mathematics. Prepared at the Entebbe Mathematics Workshops, African Education Program, Educational Services Inc., Watertown, Massachusetts (November, 1963).

Rogers, Carl R. (1959), Client-Centered Therapy. Houghton Mifflin Co., Boston.

----------. (1961), On Becoming a Person. Houghton Mifflin.

Ross, Arnold E. (1962), "Teaching Through Discovery," New York State Mathematics Teachers Journal, Vol. XII, No. 2 (April, 1962), pp. 57 - 59.

Sanders, W.J. (1964), "The Use of Models in Mathematics Instruction," The Arithmetic Teacher, Vol. 11, No. 3 (March, 1964), pp. 157 - 165.

Sawyer, W. Warwick. (1964), Vision in Elementary Mathematics. Penguin Books, Baltimore, Maryland.

School Mathematics Study Group (SMSG). (1961), Introduction to Matrix Algebra. Yale University Press.

Schwab, Joseph J. (1964), "The Teaching of Science as Enquiry," in J.J. Schwab and Paul Brandwein, The Teaching of Science, Harvard University Press, pp. 1 - 103.

Smedslund, Jan. (1964), "Internal Necessity and Contradiction in Children's Thinking," Journal of Research in Science Teaching, Vol. 2, No. 3 (September, 1964), pp. 220 - 221.

Snyder, Henry D. (1964), "An Impromptu Discovery Lesson in Algebra," The Mathematics Teacher (October, 1964), pp. 415 - 416.

Snygg, Donald. (1963), "A Learning Theory for Curricular Change," Chapter 11 in Using Current Curriculum Development. Association for Supervision and Curriculum Development NEA, 1201 Sixteenth St., N.W., Washington 36, D.C. pp. 109 - 115.

Tolman, Edward Chace. (1958), "Cognitive Maps in Rats and Men," Behavior and Psychological Man, University of California Press, Chapter 19.

Torrance, E. Paul (1963), "Creativity," Department of Classroom Teachers, American Educational Research Association of the National Education Association. Research Pamphlet Series, "What Research Says to the Teacher," No. 28 (April, 1963).

Whitehead, Alfred North. (1929), Aims of Education. Mentor Books, Macmillan Co., New York.

White, Robert W. (1959), "Motivation Reconsidered: The Concept of Competence," Psychological Review, Vol. 66, No. 5 (September, 1959), pp. 297 - 333. A publication of the American Psychological Associaiton, Inc.

Wilson, Eugene S. (1963), "Amherst Dean of Admissions Seeks Questing Quotient (QQ) Instead of IQ," Insight (Science Research Associate) Vol. 3, No. 2 (Spring, 1963), p. 1 ff.

D. The following are more general publications which have a broad relevance to the Project's activities.

Alexander, William M. (1963), "Curriculum Planning and Development: A Review of Reviews," Chapter X in Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963), pp. 330 - 335.

Allen, Layman E. "Equations -- The Game of Creative Mathematics." Wff 'n Proof Games, New Haven, Connecticut, Box 71.

Atkin, J. Myron. (1964), "Science in the Elementary School," Chapter I in Review of Educational Research, Vol. XXXIV, No. 3 (June, 1964), Natural Sciences and Mathematics, pp. 263 - 272.

Ausubel, David P. Implications of Pre-adolescent and Early Adolescent Cognitive Development for Secondary School Teaching (mimeographed). Bureau of Educational Research, University of Illinois, Urbana, Illinois.

----------. (1963), "Some Psychological and Educational Limitations of Learning by Discovery," New York State Mathematics Teachers Journal, Vol. XIII, No. 3 (June, 1963), pp. 90 - 108.

----------. (1964), "Some Psychological and Educational Limitations of Learning by Discovery," The Arithmetic Teacher, Vol. 11, No. 5 (May, 1964), pp. 290 - 302.

Balch, John. (1964), "The Influence of the Evaluating Instrument on Students' Learning," American Educational Research Journal, Vol. 1, No. 3 (May, 1964), pp. 169 - 182.

Bartlett, F. C. (1964), Remembering: A Study in Experimental and Social Psychology. Cambridge University Press.

Belanger, Maurice. (1964), "Methodology of Educational Research in Science and Mathematics," Chapter X of Review of Educational Research, Vol. XXXIV, No. 3 (June, 1964), pp. 374 - 390.

Boring, Edwin G. (1963), History, Psychology, and Science (edited by Watson and Campbell). Wiley.

Bowers, Norman D. (1963), "Psychological Forces Influencing Curriculum Decisions," Chapter IV of Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963), pp. 268 - 277.

Briggs, Leslie J. and David Angell. (1964), "Programed Instruction in Science and Mathematics," Chapter IX of Review of Educational Research, Vol. XXXIV, No. 3 (June, 1964), pp. 354 - 373.

Briggs, Leslie J. and Nancy Russell Hamilton. (1964), "Meaningful Learning and Retention: Practice and Feedback Variables," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 545 - 559.

Brooks, Sarah. (1962), "Teaching the New Curricula in Senior High," New York State Mathematics Teachers Journal, Vol. XII, No. 3 (June, 1962), pp. 114 - 115.

Brown, Roger. (1965), Social Psychology. The Free Press, Macmillan Co.

Buxton C. Lee. (1962), A Study of Psychophysical Methods for the Relief of Childbirth Pain. Saunders. Especially pp. 5 and 6, concerning observation, methodology, experience and knowledge.

Callahan, Raymond E. (1962), Education and the Cult of Efficiency. University of Chicago Press.

----------. (1964), Changing Conceptions of the Superintendency in Public Education 1865 - 1964. The New England School Development Council, Longfellow Hall, Appian Way, Cambridge, Massachusetts 02138.

Callaway, Rolland and Gerald T. Gleason. (1963), "Teaching," Chapter VI of Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963), pp. 293 - 300.

Cass, James. (1964), "Has Teaching Become a Profession Yet," Education Age, Vol. 1, No. 2 (November - December, 1964), pp. 5 - 7.

Churchill, Eileen. (1961), Counting and Measuring. Routledge and Kegan Paul, London, England.

Committee of the College and University examiners. (1956), Taxonomy of Educational Objectives, I: Cognitive Domain. Benjamin S. Bloom, editor, Longmans, Green, and Co., New York.

Conant, James Bryant. (1965), Two Modes of Thought. Pocket Books, Inc. (paperback).

Davis, O.L., Jr. (1963), "Organized Knowledge Influencing Curriculum Decisions," Chapter II of Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963) pp. 245 - 253.

Dessart, Donald J. (1964), "Mathematics in the Secondary School," Chapter IV in Review of Educational Research, Vol. XXXIV, No. 3 (June, 1964), pp. 298 - 312.

Dick, Walter. (1965), "The Development and Current Status of Computer-Based Instruction," American Educational Research Journal, Vol. 2, No. 1 (January, 1965), pp. 41 - 54.

Dienes, Z.P. (1962), "Mathematics Learning as Appreciation of Structure," New York State Mathematics Teachers Journal, Vol. XII, No. 4 (October, 1962), pp. 151 - 157.

Dobzhansky, Theodosius. (1962), Mankind Evolving. Yale University Press.

Elliott, David L. and Arthur W. Foshay. (1963), "Chart or Charter: Recent Developments in Educational Discourse," Chapter I in Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963), pp. 233 - 244.

Ennis, Robert H. (1964), "Operational Definitions," American Educational Research Journal, Vol. 1, No. 3 (May, 1964), pp. 183 - 201.

Erikson, Erik H. (1959), "Identity and the Life Cycle," Psychological Issues, Vol. 1, No. 1, pp. 18 - 171.

Fitzgerald, William M. (1963), Self-Selected Mathematics Learning Activities, Cooperative Research Project No. 2047, U.S. Office of Education (June 15, 1963 - December 14, 1964).

Fitzsimons, Lucille. (1964), "Kindergarten Mathematics," The Arithmetic Teacher, Vol. 11, No. 1 (January, 1964), pp. 33 - 35.

Flavell, John H. (1963), The Developmental Psychology of Jean Piaget. Van Nostrand Co., Inc., Princeton, New Jersey.

Fletcher, T.J. (1963), "Models of Many-Valued Logics," The American Mathematical Monthly, Vol. 70, No. 4, pp. 381 - 391.

Fraser, Dorothy M., Ole Sand, et al. (1963), Deciding What to Teach. Project on Instruction, NEA.

Frazier, Alexander. (1964), "Making Use of National Curriculum Studies," Educational Leadership, Vol. 22, No. 2 (November, 1964), pp. 101 - 131.

Friedenberg, Edgar Z. (1962), The Vanishing Adolescent. Dell Publishing Company (paperback).

Gage, N.L. (1963), Handbook of Research on Teaching, Rand-McNally, pp. 247 - 328.

Gagné, Robert M. (1965), The Conditions of Learning. Holt, Rinehart and Winston.

Gallagher, James J. (1964), "Meaningful Learning and Retention: Intrapersonal Cognitive Variables," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 499 - 512.

Gardner, Eric F., et al. (1962), "Educational and Psychological Testing," Review of Educational Research, Vol. XXXII, No. 1 (February, 1962).

Getzels, Jacob W. and Keith Elkins. (1964), "Perceptual and Cognitive Development," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 559 - 573.

Gordon, Ira J. (1962), Human Development. Harper and Row.

Gottlieb, David. (1964), "Sociology of Education," Chapter IV of Review of Educational Research, Vol. XXXIV, No. 1 (February, 1964), pp. 62 - 70.

Goutard, Madeleine. (1964), Mathematics and Children. Education Explorers Limited, Reading, England.

Graham, W.A. (1964), "Individual Teaching of Fifth and Sixth-Grade Arithmetic," The Arithmetic Teacher, Vol. 11, No. 4 (April, 1964), pp. 223 - 234.

Hadamard, Jacques. (1944), The Psychology of Invention in the Mathematical Field. Dover Publications.

Heath, Robert (editor). (1964), New Curricula. Harper and Row, New York.

Hechinger, Fred M. (1965), "Translating Higher Pay Into Better Teachers," New York Times (Monday, August 16, 1965).

Hilgard, Ernest R. (editor). (1964), Theories of Learning and Instruction. Sixty-Third Yearbook, Part I. National Society for the Study of Education. University of Chicago Press.

Hlavaty, Julius H. (1962), "Mathematics Education in Transition," New York State Mathematics Teachers Journal, Vol. XII, No. 3 (June, 1962), pp. 86 - 92.

Ho, Wai-Ching and Agatha Konieczynski. (1965), Reliabilities and Other Anaylses of the Greater Cleveland Mathematics Tests. Education Research Council of Greater Cleveland. (July, 1965).

Hunt, J. McV. (1961), Intelligence and Experience. Ronald Press.

Inbody, Donald. (1965), "Helping Parents Understand New Mathematics Programs," The Arithmetic Teacher, Vol. 11, No. 8 (December, 1964), pp. 530 - 537.

Jackson, Philip W. and Nina Strattner. (1964), "Meaningful Learning and Retention: Noncognitive Variables," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 513 - 529.

Kagan, Jerome. "The Choice of Models: A Developmental Analysis of Continuity, Discontinuity, and Conflict in Human Behavior " (mimeographed). Department of Social Relations, Harvard University.

Kaufman, Bel. (1964), Up the Down Staircase. Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Kloepfel, Robert E. (1962), "An Experimental Program in a Small Central School," New York State Mathematics Teachers Journal, Vol. XII, No. 3 (June, 1962), pp. 116 - 119.

Klohr, Paul R. and Jack R. Frymeir. (1963), "Curriculum Development: Dynamics of Change," Chapter VIII of Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963), pp. 304 - 321.

Koch, Sigmund. (1964), "Psychology and Emerging Conceptions of Knowledge as Unitary," in Behaviorism and Phenomenology, edited by T.W. Wann, University of Chicago Press.

Kooi, Beverly Y. and Richard E. Schutz. (1965), "A Factor Analysis of Classroom-Disturbance Intercorrelations," American Educational Research Journal, Vol. 2, No. 1 (January, 1965), pp. 37 - 40.

Krathwohl, David R., et al. (1964), Taxonomy of Educational Objectives, II: Affective Domain. David McKay.

Krumboltz, John D. (1964), "The Nature and Importance of the Required Response in Programed Instruction," American Educational Research Journal, Vol. 1, No. 4 (November, 1964), pp. 203 - 209.

Lucow, William H. (1964), "An Experiment with the Cuisenaire Method in Grade Three," American Educational Research Journal, Vol. 1, No. 3 (May, 1964), pp. 159 - 167.

Macdonald, James B. and James D. Raths. (1963), "Curriculum Research: Problems, Techniques, and Prospects," Chapter IX in Review of Educational Research, Vol. XXXIII, No. 3 (June, 1963), pp. 322 - 329.

Mayo, Samuel T., et al. (1965), "Educational and Psychological Testing," Review of Educational Research, Vol. XXXV, No. 1 (February, 1965).

McClelland, David C. (1961), The Achieving Society. Van Nostrand.

McCullers, John C. and Walter T. Plant. (1964), "Personality and Social Development: Cultural Influences," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 599 - 610.

McDonald, Frederick J. (1964), "Meaningful Learning and Retention: Task and Method Variables," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 530 - 544.

Meyer, Donald L. (1964), "A Bayesian School Superintendent," American Educational Research Journal, Vol. 1, No. 4 (November, 1964), pp. 219 - 228.

Miles, Matthew B. (editor). (1964), Innovation in Education, Bureau of Publications, Teachers College, Columbia University.

Millman, Jason and Mauritz Johnson, Jr. (1964), "Relation of Section Variance to Achievement Gains in English and Mathematics in Grades 7 and 8," American Educational Research Journal, Vol. 1, No. 1 (January, 1964), pp. 47 - 51.

Montessori, Maria and Nancy McCormick Rambusch. (1965), Dr. Montessori's Own Handbook. Schocken (based upon a publication by Montessori in 1914).

Montessori, Maria and John J. McDermott. (1965), Spontaneous Activity in Education. Schocken (based upon a publication by Montessori in 1917).

Moss, Ruth and Lola May. (1965), "The New Math -- For Adults Only," Chicago Tribune Sunday Magazine (beginning February 21, 1965 and ending May 16, 1965).

Mosteller, Frederick, G.B. Thomas, Jr., and R.E.K. Rourke. (1961), Probability with Statistical Applications. Addison-Wesley, Inc.

Mueller, Marvin M. and N.R. Hanson. (1965), "Two Notes on Force-Free Motion," Science, Vol. 149, No. 3688 (September 3, 1965), pp. 1048 - 1049. This involves some interesting questions of scientific model-making, in regard to one of the most classical of all abstract models.

Murphy, Lois., et al. (1962), The Widening World of Childhood. Basic Books.

Newman, Claire M. (1963), "An Objection to Objectivity," New York State Mathematics Teachers Journal, Vol. XIII, No. 3 (June, 1963), pp. 127 - 128.

Orlich, Donald C. (1964), "The Dawn of Scientific Epistemology: 1564 - 1964," Journal of Research in Science Teaching, Vol. 2, Issue 2 (June, 1964), pp. 95 - 99.

Parsons, Cynthia. (1965), " 'New' Study Methods Tailored to Fit," No. 7 of "What's Right With U.S. Public Schools?" Christian Science Monitor, Sec. 2 (August 3, 1965), pp. 1.

Perkins, Hugh V. (1964), "A Procedure for Assessing the Classroom Behavior of Students and Teachers," American Educational Research Journal, Vol. 1, No. 4 (November, 1964), pp. 249 - 260.

----------. (1965), "Classroom Behavior and Underachievement," American Educational Research Journal, Vol. 2, No. 1 (January, 1965), pp. 1 - 12.

Perkus, Robert, (1963), "Notes on the International Congress of Mathematics -- Stockholm, 1962," New York State Mathematics Teachers Journal, Vol. XIII, No. 4 (October, 1963), pp. 154 - 163.

Photographing a Class of Students in Action at the ESI Studio. (1965), Educational Services, Inc., Watertown, Massachusetts.

Polanyi, Michael. (1958), Personal Knowledge. University of Chicago Press.

Redl, Fritz and David Wineman. (1957), Children Who Hate, Free Press, Glencoe, New York.

Rogers, Hartley, Jr. (1963), "An Example in Mathematical Logic," American Mathematical Monthly, Vol. 70, No. 9, pp. 929 - 945.

Rosenbloom, Paul C. (editor). (1964), Modern Viewpoints in the Curriculum. McGraw-Hill.

Ross, R. (1964), "A Description of Twenty Arithmetic Underachievers," The Arithmetic Teacher, Vol. 11, No. 4 (April, 1964), pp. 235 - 241.

Rudin, Stanley A. (1965), "The Personal Price of National Glory," Trans-action (September - October, 1965), Vol. 2, No. 6, pp. 4 - 9.

Ryans, David G. (1963), "Assessment of Teacher Behavior and Instruction," Chapter VII of Review of Educational Research, Vol. XXXIII, No. 4 (October, 1963), pp. 415 - 441.

Sealey, L.G.W. (1961), The Creative Use of Mathematics in the Junior School. Blackwell, Oxford, England.

Shunk, William R. and Bernice Z. Goldstein. (1964), "Anthropology and Education," Chapter V in Review of Educational Research, Vol. XXXIV, No. 1 (February, 1964), pp. 71 - 84.

Smith, Lewis B. (1965), "Pegboard Geometry," The Arithmetic Teacher, Vol. 12, No. 4 (April, 1965), pp. 271 - 274.

Spaulding, Robert L. (1964), "Personality and Social Development: Peer and School Influences," Review of Educational Research, Vol. XXXIV, No. 5 (December, 1964), pp. 588 - 598.

Stanton, Alfred H. and Morris S. Schwartz. (1954), The Mental Hospital. Basic Books. Their attempt to study a mental hospital as a social institution has direct relevance to the task of studying the school as a social institution.

Starke, Emory. (1964), "Contemporary Mathematics," The New Jersey Mathematics Teacher, Vol. XXI, No. 2 (February, 1964), pp. 23 - 26.

Taba, Hilda. (1962), Curriculum Development: Theory and Practice. Harcourt, Brace and World.

Travers, Robert M. W. and Keith Van Wagenen. (1964), "Observational Learning in a Simulated Recitation Group," American Educational Research Journal, Vol. No. 1 (January, 1964), pp 26 - 34.

Veatch, Jeannette. (1959), Individualizing Your Reading Program. The Putnam Series in Education. G. P. Putnam's Sons, New York.

Vygotsky, Lev Semenovich. (1962), Thought and Language (translated by Hanfmann and Vakar). M.I.T. Press.

Wann, Kenneth D., Miriam Selchen Dorn, and Elizabeth Ann Liddle. (1962), Fostering Intellectual Development in Young Children. Bureau of Publications, Teachers College, Columbia University, New York.

Weaver, J. Fred. and E. Glenadine Gibb. (1964), "Mathematics in the Elementary School," Chapter II in Review of Educational Research, Vol. XXXIV, No. 3 (June, 1964), Natural Sciences and Mathematics, pp. 273 - 285.

Weidman, Donald R. (1965), "Emotional Perils of Mathematics," Science, Vol. 149, No. 3688 (September 3, 1965), p. 1048.

Weinberg, Alvin M. (1965), "But is the Teacher Also a Citizen?" Science, Vol. 149,No. 3684 (August 6, 1965), pp. 601 - 606.

Wernick, William. (1964), "An Experiment in Teaching Mathematics to Children," The Arithmetic Teacher, Vol. 11, No. 3 (March, 1964), pp. 150 - 156.

Wertheimer, Max. (1959), Productive Thinking. Harper and Row.

Wirtz, Robert W., Botel and Sawyer. Mathematics Workshop. Encyclopedia Brittannica Films, 425 North Michigan Avenue, Chicago 11, Illinois.

Wittrock, M. C. (1964), The Learning by Discovery Hypothesis, U.C.L.A. (mimeographed) (December, 1964).